Year 12 Mathematics Extension 2
2013 HSC ASSESSMENT TASK 1
Term 4 Week 8 2012

Name:____________________________

Teacher:__________________________

Monday 26th November
Periods 5 & 6

Set by: VUL

- Attempt **all** questions.
- All questions are of equal value.
- Marks may be deducted for insufficient, or illegible work.
- Only Board approved calculators (**excluding** graphic calculators) may be used
- Total possible mark is **50**
- **Begin each question on a new page.**
- **TIME ALLOWED :** 90 minutes + 2 minutes reading time.

Question 1 **Marks**

(a) Let $z = 2\sqrt{3} + i$ and $w = \sqrt{3} - i$

(i) Find $2z - \bar{w}$ in the form $x + iy$ where x and y are real. **1**

(ii) Find $\dfrac{z}{w}$ in the form $x + iy$ where x and y are real **2**

(b) Find the two square roots of $3 - 4i$ for z giving your answers in the form $x + iy$ where x and y are real. **3**

(c) (i) Express $-1 - \sqrt{3}i$ in modulus-argument form. **2**

(ii) Hence evaluate $\left(-1 - \sqrt{3}i\right)^9$ **2**

(d) On separate Argand diagrams, sketch the locus of points z such that:

(i) $\arg\left(z - 1 + i\right) = \dfrac{\pi}{2}$ **2**

(ii) the inequalities $|z - i| \le 2$ and $1 \le \text{Im}(z) \le 2$ both hold **2**

(iii) $|z + \bar{z}| = 1$ **2**

(e) Find the three cube roots of $-8i$ in the form $x + iy$ where x and y are real. **3**

Question 1 is continued on the next page

Question 1 ***Continued*** **Marks**

(f) (i) By rationalising the numerator of $\dfrac{\sqrt{n+1}-\sqrt{n}}{1}$ prove that **2**

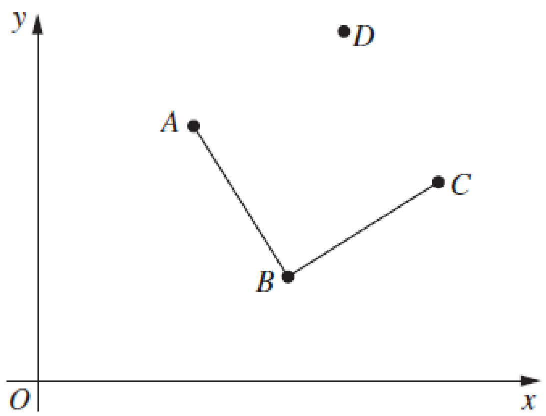
$\sqrt{n+1}-\sqrt{n} > \dfrac{1}{2\sqrt{n+1}}$.

(ii) Hence prove by mathematical induction that **4**

$\sqrt{n} > 1+\dfrac{1}{2}+\dfrac{1}{3}+...+\dfrac{1}{n}$ for $n \geq 7$

Question 2 **Start a new page.**

(a)



In the diagram the vertices of a triangle ABC are represented by the complex numbers z_1, z_2 and z_3, respectively. The triangle is isosceles and right-angled at B.

(i) Explain why $(z_1 - z_2)^2 = -(z_3 - z_2)^2$. **1**

(ii) Suppose D is the point such that $ABCD$ is a square. Find the complex number, expressed in terms of z_1, z_2 and z_3, that represents D. **2**

(b) (i) Sketch the locus of the complex number $z = x + iy$

where $\arg[z-1] - \arg[z+1] = \dfrac{\pi}{4}$. **1**

(ii) Find the Cartesian equation of the locus described in part (i) **1**

(iii) Give the range of the locus found in part (ii). **1**

Question 2 is continued on the next page.

Question 2 *Continued*

(c) If $z = \cos\theta + i\sin\theta$:

(i) Show that $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$ **2**

(ii) Use the binomial theorem to expand $\left(z - \dfrac{1}{z}\right)^5$ **1**

(iii) Hence express $\sin^5\theta$ in terms of $\sin n\theta$ **2**

(d) (i) If ω is a complex root of $z^5 - 1 = 0$ with least positive argument, show that $\omega^2, \omega^3, \omega^4$ are the other complex roots. **2**

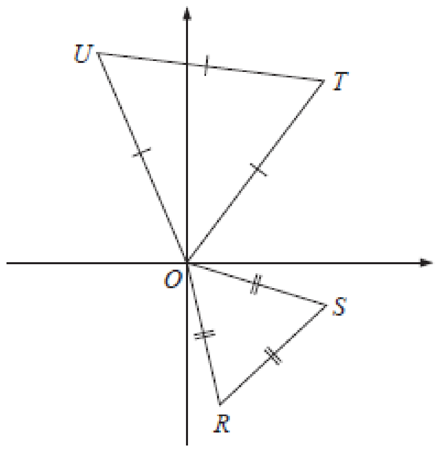
(ii) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ **1**

(iii) Plot all the roots of $z^5 - 1 = 0$ on an argand diagram. **1**

(iv) Express $z^4 + z^3 + z^2 + z + 1$ as a product of two quadratic factors. **3**

(v) Prove that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$ **2**

(e)



The diagram shows points $O, R, S, T,$ and U in the complex plane. These points correspond to the complex numbers $0, r, s, t,$ and u respectively. The triangles ORS and OTU are equilateral. Let $\omega = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$.

(i) Explain why $u = \omega t$. **1**

(ii) Find the complex number r in terms of s. **1**

(iii) Using complex numbers, show that the lengths of RT and SU are equal. **3**

End of Assessment Task

Suggested Solution (s)	Comments

Question 1

a) i) $2z - \bar{w} = 3\sqrt{3} + i$ ✓

ii) $\frac{z}{w} = \frac{2\sqrt{3}+i}{\sqrt{3}-i} \times \frac{\sqrt{3}+i}{\sqrt{3}+i}$ ✓

$= \frac{5}{4} + \frac{3\sqrt{3}}{4} i$ ✓

b) let $(x+iy)^2 = 3-4i$

$\therefore x^2 - y^2 + 2xyi = 3-4i$

$x^2 - y^2 = 3 \qquad xy = -2$

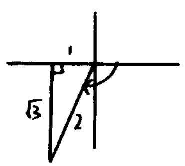
$x = \pm 2 \qquad y = \mp 1$

✓ progress

$\therefore$ square roots are

$2-i$ ✓ and $-2+i$ ✓ (one for each root)

c) i)



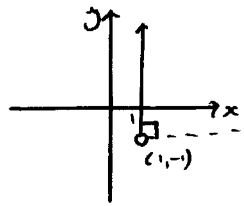
$\therefore -1-\sqrt{3}i = 2\text{cis}\left(-\frac{2\pi}{3}\right)$ ✓

ii) $(-1-\sqrt{3}i)^9 = \left[2\text{cis}\left(-\frac{2\pi}{3}\right)\right]^9$ ✓

$= 2^9 \text{cis}(-6\pi)$

$= 2^9$ or 512 ✓

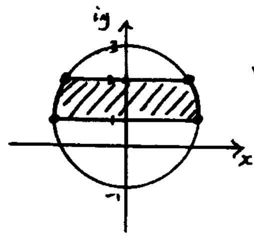
d) i) $\arg(z-(1-i)) = \frac{\pi}{2}$



✓ - ray inclined at 90°

✓ open circle at (1,-1)

ii) $|z-i| \leq 2$ and $1 \leq y \leq 2$



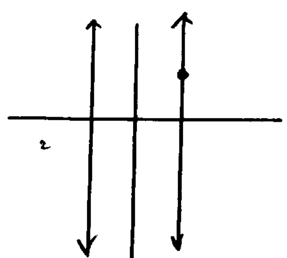
✓ circle

✓ shaded region.

iii) $|z + \bar{z}| = 1$

$|2x| = 1$

$x = \pm\frac{1}{2}$



$x = -\frac{1}{2}$ ✓ $\qquad x = \frac{1}{2}$ ✓

2013 Year 12 Mathematics Extension 2 Task 1 SOLUTIONS

Suggested Solution (s)	Comments

Question 1 Continued

e) let $-8i = 8\text{cis}\left(-\frac{\pi}{2}\right)$

$\therefore z^3 = 8\text{cis}\left(-\frac{\pi}{2}\right)$

$\therefore z_k = 8^{\frac{1}{3}} \text{cis}\left(\frac{-\frac{\pi}{2} + 2k\pi}{3}\right)$

for $k = 0, 1, 2$

So $z_0 = 2\text{cis}\left(-\frac{\pi}{6}\right) = \sqrt{3} - i$ ✓

$z_1 = 2\text{cis}\left(\frac{\pi}{2}\right) = 2i$ ✓

$z_3 = 2\text{cis}\left(\frac{7\pi}{6}\right) = -\sqrt{3} - i$ ✓

f) i) $\frac{\sqrt{n+1} - \sqrt{n}}{1} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$

$= \frac{1}{\sqrt{n+1}+\sqrt{n}}$ ✓

So $\sqrt{n+1} - \sqrt{n} - \frac{1}{2\sqrt{n+1}}$

$= \frac{1}{\sqrt{n+1}+\sqrt{n}} - \frac{1}{2\sqrt{n+1}}$

$= \frac{\sqrt{n+1} - \sqrt{n}}{(\sqrt{n+1}+\sqrt{n})(2\sqrt{n+1})} > 0$ ✓

as $\sqrt{n+1} - \sqrt{n} > 0$ and $(\sqrt{n+1}+\sqrt{n})(2\sqrt{n+1}) > 0$

$\therefore \sqrt{n+1} - \sqrt{n} > \frac{1}{2\sqrt{n+1}}$

ii) When $n = 7$

$\sqrt{n} - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$

$= \sqrt{7} - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{7}\right)$

$= \sqrt{7} - \frac{363}{140}$

$\doteq 0.0529 > 0$ from calc. ✓

Assume statement true for $n=k$

ie $\sqrt{k} > 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$

When $n = k+1$,

$\sqrt{n} - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$

$= \sqrt{k+1} - \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} + \frac{1}{k+1}\right)$

$> \sqrt{k+1} - \left(\sqrt{k} + \frac{1}{k+1}\right)$ by assumption ✓

$= \sqrt{k+1} - \sqrt{k} - \frac{1}{k+1}$

$> \frac{1}{2\sqrt{k+1}} - \frac{1}{k+1}$ from part i) ✓

$= \frac{\sqrt{k+1}}{2(k+1)} - \frac{1}{k+1}$ rationalising denominator ✓

$= \frac{\sqrt{k+1} - 2}{2(k+1)} > 0$ for $k \geq 7$

$\therefore$ If statement true for $n=k$, then proved true for $n=k+1$,

Since true for $n=7$,

$\therefore$ True for $n = 8, 9, 10 \ldots$

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Question 2

(a) i) $\overrightarrow{BA} = i \times \overrightarrow{BC}$ anticlockwise rotation of $90°$, squaring we have ✓

$(z_1 - z_2)^2 = -(z_3 - z_2)^2$

as $\overrightarrow{BA} = z_1 - z_2$

$\overrightarrow{BC} = z_3 - z_2$

ii) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$ ✓

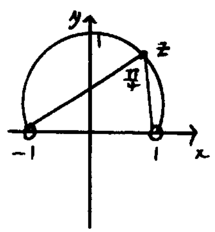
$= \overrightarrow{OA} + \overrightarrow{BC}$

as $\overrightarrow{BC} = \overrightarrow{AD}$ opp. sides of square

$\therefore \overrightarrow{OD} = z_1 + z_3 - z_2$ ✓

alternative solutions at end of document

b) i)



✓

ii) $x^2 + (y-1)^2 = 2$ ✓

iii) $0 < y \le 1 + \sqrt{2}$ ✓

c) i) $z^n = \cos n\theta + i \sin n\theta$ ✓

$z^{-n} = \cos n\theta - i \sin n\theta$

$\therefore z^n - z^{-n} = 2i \sin n\theta$ ✓

ii) $(z - \frac{1}{z})^5$ ✓

$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$

iii) Let $n = 1$, ✓

$(2i\sin\theta)^5 = z^5 - \frac{1}{z^5} - 5(z^3 - \frac{1}{z^3}) + 10(z - \frac{1}{z})$

$32 i \sin^5\theta = 2i\sin 5\theta - 10 i \sin 3\theta + 20 i \sin\theta$

$\div 32i$

$\therefore \sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$ ✓

d) i) The roots of $z^5 - 1 = 0$ are equally spaced $\frac{2\pi}{5}$ radians apart on the unit circle starting with $z_0 = 1$ ✓ or equiv.

So let $z_1 = \text{cis}\frac{2\pi}{5} = \omega$

$\therefore z_2 = \text{cis}\frac{4\pi}{5} = (\text{cis}\frac{2\pi}{5})^2 = \omega^2$

Similarly $z_3 = \text{cis}\frac{6\pi}{5} = \omega^3$

$z_4 = \text{cis}\frac{8\pi}{5} = \omega^4$ ✓ or equiv.

ii) ✓

$\therefore 1 + \omega + \omega^2 + \omega^3 + \omega^4$

$=$ sum of roots

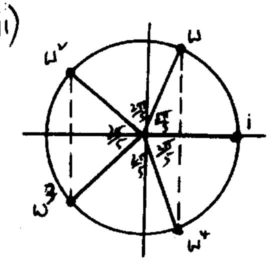
$= -\frac{b}{a}$ of $z^5 - 1 = 0$

$= \frac{-0}{1} = 0$

2013 Year 12 Mathematics Extension 2 Task 1 SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

iii)



✓

iv) $z^5 - 1$ ✓ for progress

$= (z-\omega)(z-\omega^4)(z-\omega^2)(z-\omega^3)(z-1)$

$= (z-\omega)(z-\bar{\omega})(z-\omega^2)(z-\bar{\omega}^2)(z-1)$

$= [z^2 - 2\text{Re}(\omega)z + |\omega|^2][z^2 - 2\text{Re}(\omega^2)z + |\omega^2|^2]$

$= (z-1)(z^2 - 2\cos\frac{2\pi}{5}z + 1)(z^2 - 2\cos\frac{4\pi}{5}z + 1)$

✓ ✓ (One for each quadratic factor)

Since

$z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$

$\therefore z^4 + z^3 + z^2 + z + 1$

$= (z^2 - 2\cos\frac{2\pi}{5}z + 1)(z^2 - 2\cos\frac{4\pi}{5}z + 1)$

v) ω and ω^4 are roots of $z^2 - 2\cos\frac{2\pi}{5}z + 1$ ✓ significant progress

$\therefore \omega + \omega^4 = \frac{-b}{a} = 2\cos\frac{2\pi}{5}$ — ①

Similarly ω^2 and ω^3 are roots of $z^2 - 2\cos\frac{4\pi}{5}z + 1$

$\therefore \omega^2 + \omega^3 = \frac{-b}{a} = 2\cos\frac{4\pi}{5}$ — ②

So ① + ②:

$\omega + \omega^2 + \omega^3 + \omega^4 = 2(\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5})$

but $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$

$\therefore -1 = 2(\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5})$

$\therefore \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ ✓

e) i) Since ΔTOU is equilateral

$\angle TOU = \frac{\pi}{3}$ ✓

Now $\overrightarrow{OU} = \overrightarrow{OT} \times \text{cis}\frac{\pi}{3}$ (clockwise rotation of $\frac{\pi}{3}$)

$\therefore u = \omega t$

ii) Similarly

$\overrightarrow{OR} = \overrightarrow{OS} \times \text{cis}(-\frac{\pi}{3})$ (anticlockwise rotation of $-\frac{\pi}{3}$)

$\therefore r = \bar{\omega} s$ or equivalent ✓

$= \frac{s}{\omega}$

$r = |s| \text{cis}(-\frac{\pi}{3} + \arg s)$

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Question 2 Continued iii) Since $\vec{SU} = u - s$ and $\vec{RT} = t - r$ $\omega \times \vec{RT} = \omega t - \omega r$ $= u - s$ as if $r = \bar{\omega} s$ $s = \omega r$ Now $\lvert \omega \times \vec{RT} \rvert = \lvert u - s \rvert$ $\therefore \lvert \omega \rvert \times \lvert \vec{RT} \rvert = \lvert u - s \rvert$ but $\lvert \omega \rvert = 1$ $\therefore 1 \times \lvert RT \rvert = \lvert u - s \rvert$ $\lvert RT \rvert = \lvert u - s \rvert$ $= \lvert SU \rvert$ as required * See attached document for alternate solutions	✓ ✓ ✓		

2013 Year 12 Mathematics Extension 2 Task 1 SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
ALTERNATIVE SOLUTIONS to QUESTION 2(a)(ii) Let $D(z_4)$. **VERSION 1** $\vec{CD} = -i\vec{CB}$ $z_4 - z_3 = -i(z_2 - z_3)$ $z_4 = z_3 + i(z_3 - z_2)$ OR $z_4 = (1+i)z_3 - iz_2$. **VERSION 2** $\vec{BD} = \vec{BC} + \vec{BA}$ (parallelogram of vectors) $= (z_3 - z_2) + (z_1 - z_2)$ i.e. $\vec{BD} = z_1 + z_3 - 2z_2$ AND $\vec{BD} = \vec{OD} - \vec{OB}$ So $z_4 - z_2 = z_1 + z_3 - 2z_2$ $\Rightarrow z_4 = z_1 + z_3 - z_2$. **VERSION 3** $\vec{AD} = i\vec{AB}$ $= i(z_2 - z_1)$ i.e. $z_4 - z_1 = i(z_2 - z_1)$ $\Rightarrow z_4 = z_1 + i(z_2 - z_1)$ OR $z_4 = (1-i)z_1 + iz_2$.		**VERSION 4** $\vec{OD} = \vec{BC} \times \underbrace{\sqrt{2}}_{\text{dilation factor}} \times \overbrace{\text{cis}\,\frac{\pi}{4}}^{\text{rotation factor}}$ $= (z_3 - z_2) \times \sqrt{2} \times \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$ $= (1+i)(z_3 - z_2)$ OR $z_4 = (z_3 - z_2) + i(z_3 - z_2)$. **VERSION 5** $\vec{BD} = i\vec{AC}$ $z_4 - z_2 = i(z_3 - z_1)$ $\Rightarrow z_4 = z_2 + i(z_3 - z_1)$ OR $z_4 = z_2 + iz_3 - iz_1$ **VERSION 6** $\vec{AD} = i\vec{AB} \Rightarrow z_4 - z_1 = i(z_2 - z_1)$ - (i) & $\vec{CD} = i\vec{BC} \Rightarrow z_4 - z_3 = i(z_3 - z_2)$ - (ii) (i) + (ii) gives: $2z_4 - z_1 - z_3 = i(z_3 - z_1)$ $\Rightarrow 2z_4 = (z_1 + z_3) + i(z_3 - z_1)$ $\Rightarrow z_4 = \frac{1}{2}(1-i)z_1 + \frac{1}{2}(1+i)z_3$ OR $z_4 = \frac{1}{2}(z_1 + z_3) + \frac{1}{2}i(z_3 - z_1)$	