Year 12 Mathematics Extension 2	Name:____________________
2014 HSC ASSESSMENT TASK 1	Teacher:____________________
Term 4 Week 10 2013	

Setter: VUL **Wed 4th December Periods 4 and 5**

- Attempt **all** questions.
- All questions are of equal value.
- Marks may be deducted for insufficient, or illegible work.
- Only Board approved calculators may be used
- Total possible mark is **50**
- **Begin each question on a new page.**
- **TIME ALLOWED :** 90 minutes + 2 minutes reading time.

Question 1 **Marks**

(a) Let $z=\sqrt{3}-i$ and $w=i-1$

(i) Find $z-\overline{w}$ in the form $x+iy$ where x and y are real. **1**

(ii) Find $\dfrac{z}{w}$ in the form $x+iy$ where x and y are real **2**

(iii) Write z and w in modulus-argument form **2**

(iv) Hence or otherwise show that $\dfrac{(\sqrt{3}-i)^{30}}{(i-1)^{50}}$ is purely imaginary **3**

(b) Solve $2z^2-(3+i)z+2=0$ **4**

(c) Find the Cartesian equations and give a geometric description of the locus of points z such that:

(i) $|z+2|=|z-i|$ **2**

(ii) $|z+2|=2|z-i|$ **2**

(d) Sketch the locus of points so z so that the inequalities $|z+\overline{z}|\le 2$ and $\dfrac{\pi}{4}\le \arg(z-i)\le \dfrac{3\pi}{4}$ hold simultaneously **3**

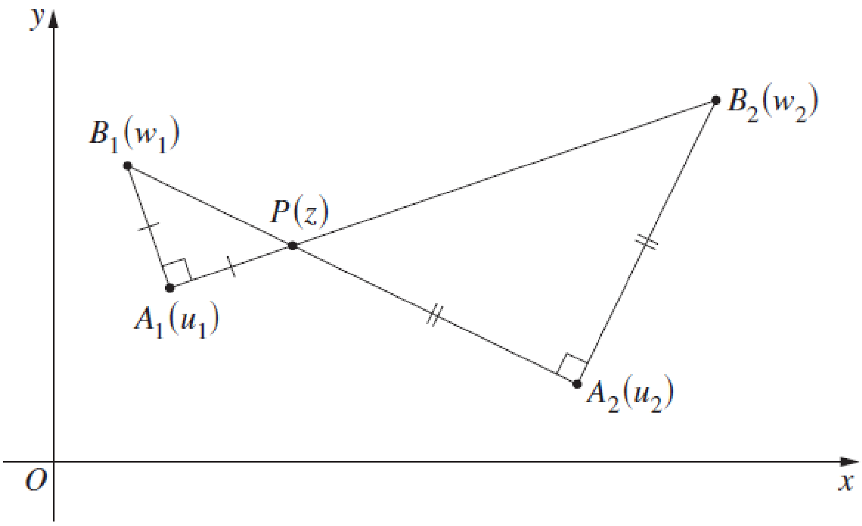
(e) On an Argand diagram plot the position of a complex number z with $0<\operatorname{Re}(z)<1$, $0<\operatorname{Im}(z)<1$ and $\dfrac{\pi}{4}<\arg(z)<\dfrac{\pi}{2}$. Then plot z^2 on the same Agrand diagram. **2**

(f) Prove by mathematical induction that $\displaystyle\sum_{r=1}^{n}\frac{1}{r^2}\le 2-\frac{1}{n}$ for all integers $n\ge 1$ **4**

Question 2 **Start a new page** **Marks**

(a) On the Argand diagram the points A_1 and A_2 correspond to the distinct complex numbers u_1 and u_2 respectively. Let P be a point corresponding to a third complex number z.

Points B_1 and B_2 are positioned so that $\triangle A_1PB_1$ and $\triangle A_2B_2P$, labelled in an anti-clockwise direction, are right-angled and isosceles with right angles at A_1 and A_2, respectively. The complex numbers w_1 and w_2 correspond to B_1 and B_2, respectively.



(i) Explain why $w_1 = u_1 + i(z - u_1)$. **2**

(ii) Find the locus of the midpoint of B_1B_2 as P varies. **2**

(b) If $z = \cos\theta + i\sin\theta$:

(i) Show that $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$ **2**

(ii) By expanding $\left(z - \dfrac{1}{z}\right)^3$ express $\sin^3\theta$ in terms of $\sin n\theta$ **3**

(c) (i) Given that ω is a complex cube root of unity, show that ω^2 is the other complex root. **2**

(ii) Plot the three cube roots of unity on an Argand diagram **1**

(iii) Show that $1 + \omega + \omega^2 = 0$ **1**

(iv) Simplify $\left(7 + 9\omega^4 + 7\omega^{-1}\right)^6$ **2**

Question 2 continued on the next page

Question 2 Continued

(d) (i) Solve $z^6 + 1 = 0$ over the set of complex numbers. **3**

(ii) Hence or otherwise express $z^6 + 1$ as product of real quadratic factors. **3**

(e) Let $z_2 = 1 + i$ and, for $n > 2$, let $z_n = z_{n-1}\left(1 + \dfrac{i}{|z_{n-1}|}\right)$. **4**

Use mathematical induction to prove that $|z_n| = \sqrt{n}$ for all integers $n \geq 2$.

End of Assessment Task

2014 (2013 Term4) Year 12 Mathematics Extension 2 HSC Task 1 Solutions

Suggested Solution (s)	Comments	Suggested Solution (s)

Question 1

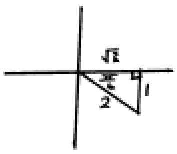
a) i) $z - \bar{w} = \sqrt{3} - i - (-1 - i)$
$= 1 + \sqrt{3}$ ✓

ii) $\frac{z}{w} = \frac{\sqrt{3} - i}{-1 + i} \times \frac{-1 - i}{-1 - i}$ ✓

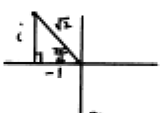
$= \frac{-\sqrt{3} - \sqrt{3}i - i - 1}{2}$

$= \frac{-\sqrt{3} - 1}{2} - \frac{(\sqrt{3} + 1)}{2} i$ ✓

iii) $z = \sqrt{3} - i$
$= 2 \operatorname{cis}\left(-\frac{\pi}{6}\right)$ ✓



$w = -1 + i$
$= \sqrt{2} \operatorname{cis}\left(\frac{3\pi}{4}\right)$ ✓



iv) $\frac{(\sqrt{3} - i)^{30}}{(i - 1)^{50}} = \frac{\left[2 \operatorname{cis}\left(-\frac{\pi}{6}\right)\right]^{30}}{\left(\sqrt{2} \operatorname{cis}\left(\frac{3\pi}{4}\right)\right]^{50}}$

$= \frac{2^{30} \operatorname{cis}(-5\pi)}{2^{25} \operatorname{cis}\left(\frac{75\pi}{2}\right)}$ by DMT ✓

$= 2^{5} \operatorname{cis}\left(-5\pi - \frac{75\pi}{2}\right)$

$= 2^{5} \operatorname{cis}\left(-\frac{85\pi}{2}\right)$ ✓

$= 32 \operatorname{cis}\left(-\frac{\pi}{2}\right)$

$= -32i$ ✓

∴ Purely Imaginary.

(b) $2z^2 - (3 + i)z + 2 = 0$

$z = \frac{3 + i \pm \sqrt{(3 + i)^2 - 4(2)(2)}}{2(2)}$

$= \frac{3 + i \pm \sqrt{-8 + 6i}}{4}$ ✓

let $(x + iy)^2 = -8 + 6i$

∴ $x^2 - y^2 + 2xyi = -8 + 6i$

∴ $x^2 - y^2 = -8 \quad xy = 3$ ✓

$x = \pm 1,\ y = \pm 3$ by inspection

∴ $z = \frac{3 + i + 1 + 3i}{4}$ or $\frac{3 + i - 1 - 3i}{4}$

$= 1 + i$ ✓ or $\frac{1}{2} - \frac{1}{2}i$ ✓

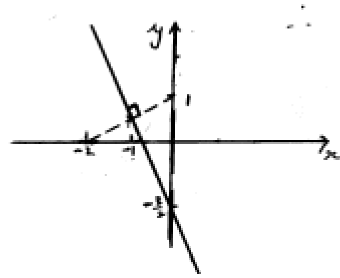
c) i) $|z + 2| = |z - i|$ is perpendicular bisector of $(-2, 0)$ and $(0, +1)$ ✓

∴ $m = -\frac{-2 - 0}{0 - (+1)}$ $\quad M\left(\frac{-2 + 0}{2}, \frac{0 + 1}{2}\right)$

$= -2$ $\quad = M\left(-1, \frac{1}{2}\right)$

∴ $y - \frac{1}{2} = -2(x + 1)$

$y = -2x - \frac{3}{2}$ ✓



2014 (2013 Term4) Year 12 Mathematics Extension 2 HSC Task 1 Solutions

Suggested Solution (s)	Comments	Suggested Solution (s)

ii) $|z+2| = 2|z-i|$

let $z = x+iy$

$\therefore |x+2+iy| = 2|x+i(y-1)|$

$\sqrt{(x+2)^2+y^2} = 2\sqrt{x^2+(y-1)^2}$

$x^2+4x+4+y^2 = 4(x^2+y^2-2y+1)$ ✓

$x^2+4x+4+y^2 = 4x^2+4y^2-8y+4$

$3x^2+3y^2-4x-8y = 0$

$x^2-\frac{4}{3}x+\frac{4}{9}+y^2-\frac{8y}{3}+\frac{16}{9} = \frac{20}{9}$

$(x-\frac{2}{3})^2+(y-\frac{4}{3})^2 = \frac{20}{9}$

$\therefore$ Circle Centre $(\frac{2}{3}, \frac{4}{3})$ radius $\frac{2\sqrt{5}}{3}$ ✓

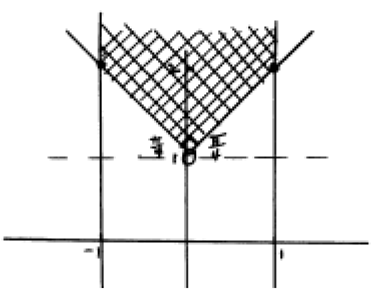
d) let $z = x+iy$

$\therefore |x+iy+x-iy| \le 2$

$|2x| \le 2$

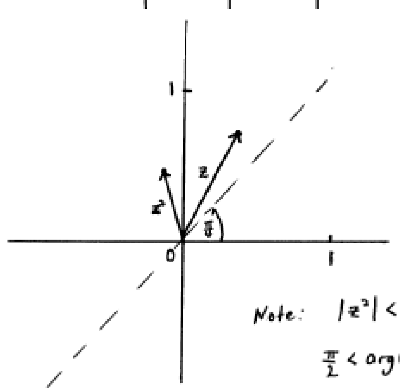
$-2 \le 2x \le 2$

$-1 \le x \le 1$ ✓



✓✓

(e)



Note: $|z^2| < |z|$ ✓

$\frac{\pi}{2} < \arg(z) <$ ✓

f)

When $n=1$, $\sum_{r=1}^{n}\frac{1}{r^2} - (2-\frac{1}{n}) = \frac{1}{1^2} - (2-\frac{1}{1})$

$= 1-1$

$= 0 \le 0$

$\therefore \sum_{r=1}^{n}\frac{1}{r^2} \le 2-\frac{1}{n}$ for $n=1$ ✓

Assume the statement is true for $n=k$, some fixed positive integer, ie $\sum_{r=1}^{k}\frac{1}{r^2} \le 2-\frac{1}{k}$

When $n=k+1$, $\sum_{r=1}^{n}\frac{1}{r^2} - (2-\frac{1}{n}) = \sum_{r=1}^{k}\frac{1}{r^2} + \frac{1}{(k+1)^2} - (2-\frac{1}{k+1})$

$\le 2-\frac{1}{k}+\frac{1}{(k+1)^2} - 2 + \frac{1}{k+1}$ ✓ by assumption

$= \frac{1}{k+1} - \frac{1}{k} + \frac{1}{(k+1)^2}$

$= \frac{k(k+1)-(k+1)^2+k}{k(k+1)^2}$

$= \frac{k^2+k-k^2-2k-1+k}{k(k+1)^2}$

$= \frac{-1}{k(k+1)^2} \le 0$ ✓ as $k>1$ and $(k+1)^2 \ge 0$ ✓

$\therefore$ If statement true for $n=k$,

It has been proved true for $n=k+1$,

Since the statement true for $n=1$,

then true for $n=2,3,4,\ldots$

$\therefore \sum_{r=1}^{n}\frac{1}{r^2} \le 2-\frac{1}{n}$ for $n \ge 1$

2014 (2013 Term4) Year 12 Mathematics Extension 2 HSC Task 1 Solutions

Question 2

a) i)

$$\begin{aligned} w_1 &= \overrightarrow{OB_1} \\ &= \overrightarrow{OA_1} + \overrightarrow{A_1B_1} \\ &= u_1 + i(z - u_1) \end{aligned}$$

but $\overrightarrow{A_1B_1} = \overrightarrow{A_1P} \times i$

$$\begin{aligned} &= [\overrightarrow{OP} - \overrightarrow{OA_1}] \times i \\ &= (z - u_1)i \end{aligned}$$

ii) Similarly

$$\begin{aligned} w_2 &= \overrightarrow{OB_2} \\ &= \overrightarrow{OA_2} + \overrightarrow{A_2B_2} \\ &= u_2 + i(u_2 - z) \end{aligned}$$

$\overrightarrow{A_2B_2} = \overrightarrow{A_2P} \times -i$

$$\begin{aligned} &= [\overrightarrow{OP} - \overrightarrow{OA_2}] \times -i \\ &= [z - u_2] \times -i \\ &= (u_2 - z)i \end{aligned}$$

Now midpoint of B_1B_2

$$\begin{aligned} &= \frac{1}{2}\left(\overrightarrow{OB_1} + \overrightarrow{OB_2}\right) \\ &= \frac{1}{2}\left[u_1 + i(z - u_1) + u_2 + i(u_2 - z)\right] \\ &= \frac{1}{2}\left(u_1 + u_2 - (u_1 - u_2)i\right) \end{aligned}$$

b) i)

$$\begin{aligned} z^n &= (\cos\theta + i\sin\theta)^n \\ &= \cos n\theta + i\sin n\theta \quad — (1) \\ z^{-n} &= \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos n\theta - i\sin n\theta \quad — (2) \end{aligned}$$

as $\cos(-A) = \cos A$, $\sin(-A) = -\sin A$

So (1)+(2):

$\therefore z^n - z^{-n} = 2i\sin n\theta$

ii)

$$\begin{aligned} \left(z - \frac{1}{z}\right)^3 &= z^3 - 3z^2\frac{1}{z} + 3z\cdot\frac{1}{z^2} - \frac{1}{z^3} \\ &= z^3 - \frac{1}{z^3} - 3\left(z - \frac{1}{z}\right) \end{aligned}$$

when $n = 1$

$(2i\sin\theta)^3 = z^3 - \frac{1}{z^3} - 3\left(z - \frac{1}{z}\right) = 2i\sin 3\theta - 3(2i\sin\theta)$

$\therefore -8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$

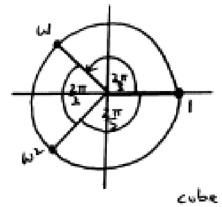
$\therefore \sin^3\theta = \frac{-1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$

c) i) The three cube roots of unity, z_0, z_1, z_2 are equally spaced $\frac{2\pi}{3}$ radians apart on unit circle with $z_0 = 1$

$\therefore z_1 = \text{cis}\frac{2\pi}{3} = w$

and $z_2 = \text{cis}\frac{4\pi}{3} = \left(\text{cis}\frac{2\pi}{3}\right)^2 = w^2$

ii)



iii) If w is a complex cube root of unity

then $w^3 = 1$

$w^3 - 1 = 0$

$(w - 1)(w^2 + w + 1) = 0$

$\therefore w^2 + w + 1 = 0$ $\quad w - 1 \neq 0$, $w \neq 1$

iv)

$$\begin{aligned} (7 + 9w^4 + 7w^{-1})^6 &= \left(7 + 9w + \frac{7}{w}\right)^6 \\ &= \left(\frac{7w + 9w^2 + 7}{w}\right)^6 \\ &= (7(1 + w + w^2) + 2w^2)^6 \\ &= (2w^2)^6 \\ &= 2^6 \cdot w^{12} \\ &= 64 \end{aligned}$$

$w^3 = 1$, $\therefore w^4 = w$

$w^3 = 1$, $w^6 = 1^2 = 1$

2014 (2013 Term4) Year 12 Mathematics Extension 2 HSC Task 1 Solutions

d) i) If $z^6 + 1 = 0$

$z^6 = -1 = \text{cis}(\pi + 2n\pi)$

$\therefore z_n = \text{cis}\left(\frac{\pi}{6} + n.\frac{\pi}{3}\right) \quad n = 0, 1, 2, \dots, 5$

$$\left.\begin{aligned} \therefore z_0 &= \text{cis}\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2} + \tfrac{i}{2} \\ z_1 &= \text{cis}\tfrac{\pi}{2} = i \\ z_2 &= \text{cis}\tfrac{5\pi}{6} = -\tfrac{\sqrt{3}}{2} + \tfrac{i}{2} \\ z_3 &= \text{cis}\tfrac{7\pi}{6} = -\tfrac{\sqrt{3}}{2} - \tfrac{i}{2} \\ z_4 &= \text{cis}\tfrac{3\pi}{2} = -i \\ z_5 &= \text{cis}\tfrac{11\pi}{6} = \tfrac{\sqrt{3}}{2} - \tfrac{i}{2} \end{aligned}\right\}$$

ii) Let
$$\begin{aligned} z^6 + 1 &= (z - z_0)(z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5) \\ &= \left(z - \text{cis}\tfrac{\pi}{6}\right)\left(z - \text{cis}\tfrac{\pi}{2}\right)\left(z - \text{cis}\tfrac{5\pi}{6}\right)\left(z - \text{cis}\tfrac{7\pi}{6}\right)\left(z - \text{cis}\tfrac{3\pi}{2}\right)\left(z - \text{cis}\tfrac{11\pi}{6}\right) \\ &= \left[z^2 - (z_0 + z_5)z + z_0 z_5\right]\left[z^2 - (z_1 + z_4)z + z_1 z_4\right]\left[z^2 - (z_2 + z_3)z + z_2 z_3\right] \\ &= \left(z^2 - \sqrt{3}z + 1\right)\left(z^2 + 1\right)\left(z^2 + \sqrt{3}z + 1\right) \end{aligned}$$

e) When $n = 2$,

$$\begin{aligned} L.H.S &= |z_n| \\ &= |z_2| \quad \text{with } z_2 = 1 + i \\ &= \sqrt{2} \\ R.H.S &= \sqrt{n} \\ &= \sqrt{2} \end{aligned}$$

$\therefore |z_n| = \sqrt{n}$ for $n = 2$

Assume the statement is true for $n = k$ some fixed positive integer

ie, $|z_k| = \sqrt{k}$

When $n = k+1$
$$\begin{aligned} |z_n| &= |z_{k+1}| \\ &= \left| z_{k+1-1}\left(1 + \frac{i}{|z_{k+1-1}|}\right) \right| \\ &= |z_k| . \left| 1 + \frac{i}{|z_k|} \right| \\ &= \sqrt{k} \left| \left(1 + \frac{i}{\sqrt{k}}\right) \right| \quad \text{by assumption} \\ &= \sqrt{k} \times \sqrt{1^2 + \left(\tfrac{1}{\sqrt{k}}\right)^2} \\ &= \sqrt{k} \times \sqrt{1 + \tfrac{1}{k}} \\ &= \sqrt{k\left(1 + \tfrac{1}{k}\right)} \\ &= \sqrt{k+1} \\ &= \sqrt{n} \quad \text{when } n = k+1 \end{aligned}$$

etc (MI conclusion).