Year 12 HSC Assessment Task 1
Mathematics Extension 2
Week 9A Monday 1st Dec 2014

Name: ____________________

Teacher: ____________________

- Attempt **ALL** questions.
- Marks may be deducted for insufficient or illegible work.
- Only Board approved calculators (**excluding** graphic calculators) may be used.
- Total possible mark is **32**.
- Begin each question on a new sheet of paper.
- **TIME ALLOWED: 40 minutes plus 2 minutes reading time.**

SECTION I – Multiple Choice (4 marks)

1 Which expression is a correct factorization of $z^3 - i$?

(A) $(z-i)(z^2+iz+1)$

(B) $(z+i)(z^2-iz-1)$

(C) $(z+1)(z-i)^2$

(D) $(z+1)^3$

2 The complex number w is a root of the equation $z^3+1=0$. Which of the following is FALSE.

(A) $\overline{w}$ is also a root

(B) $w^2+1-w=0$

(C) $\dfrac{1}{w}$ is also a root.

(D) $(w-1)^2=-1$

3 By de Moivre's theorem, the value of $(1+i)^{10}$ is:

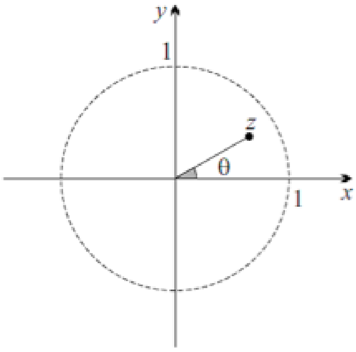
(A) purely real

(B) purely imaginary

(C) a real multiple of $(1+i)$

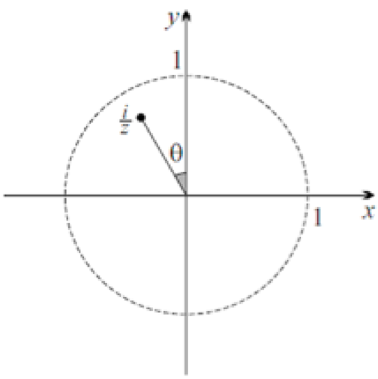
(D) an imaginary multiple of $(1+i)$

4

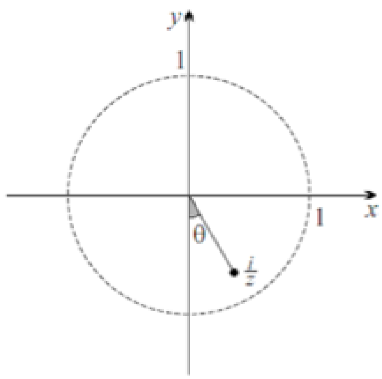


The Argand diagram above shows the complex number *z*. By considering the modulus and argument, which diagram below best represents the complex number $\dfrac{i}{z}$?

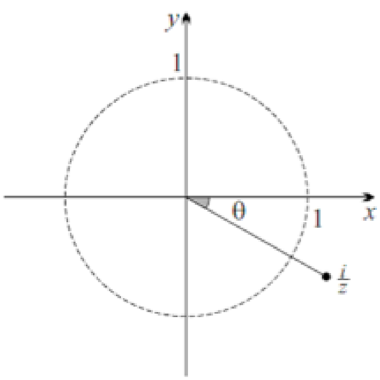
(A)



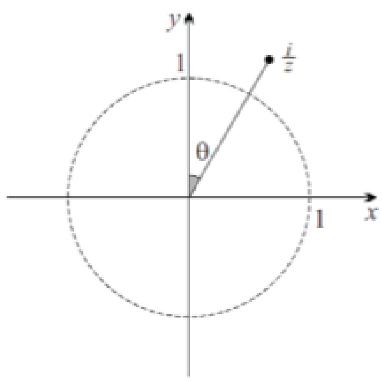
(B)



(C)



(D)



SECTION II

Question 5 (14 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the complex numbers $z = 2 + 3i$ and $w = -1 + 2i$.

Express each of the following in the form $x + iy$, where x and y are real numbers.

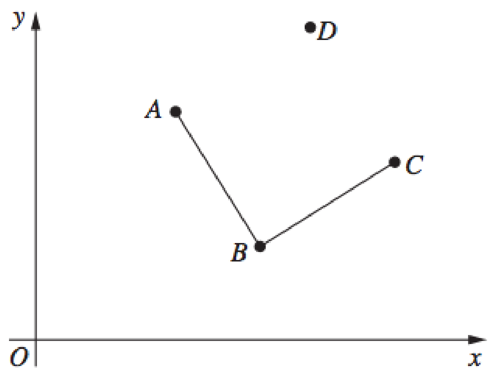
(i) $z + \overline{w}$ **1**

(ii) z^2 **1**

(iii) $\frac{z}{w}$ **1**

(b) Find the square roots of $21 - 20i$ in the form $a + ib$ where a and b are real numbers. **3**

(c)



In the diagram the vertices of a triangle ABC are represented by the complex numbers z_1, z_2 and z_3 respectively. The triangle is isosceles and right-angled at *B*.

(i) Show that $(z_1 - z_2)^2 = -(z_3 - z_2)^2$. **2**

(ii) Suppose *D* is the point such that *ABCD* is a square. Find the complex number, expressed in terms of z_1, z_2 and z_3 that represents *D*. **1**

(d) Use de Moivres Theorem, or otherwise, show that for every positive integer n **2**

$$(1+i)^n + (1-i)^n = 2(\sqrt{2})^n \cos\frac{n\pi}{4} .$$

(Do not use Mathematical Induction)

(e) Prove using the principle of Mathematical induction that, **3**

If $T_1 = 1,\ T_2 = 5$ and $T_n = 5T_{n-1} - 6T_{n-2}$ for $n \geq 3$,

show $T_n = 3^n - 2^n$ for $n \geq 1$

End of Question 5

Question 6 (14 marks) Use a SEPARATE writing booklet. **Marks**

(a) Sketch the region in the Argand diagram where **3**

$$|z - 1 - i| \le 1 \text{ and } -\frac{\pi}{4} \le Arg(z - 1 - i) \le \frac{\pi}{2}$$

(b) Find the fourth roots of $-8 + 8i\sqrt{3}$ and plot the roots on an Argand diagram. **3**

(c) The inequality $x > \ln(1+x)$ holds for all real $x > 0$. **3**
(Do NOT prove this.)

Use this result and the method of mathematical induction to prove that for all positive integers *n*,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} > \ln(n+1).$$

(d) Let the points A_1, A_2, …, A_n represent the *n*th roots of unity, $w_1, w_2, ..., w_n$, and suppose *P* represents any complex number *z* such that $|z| = 1$.

(i) Prove that $w_1 + w_2 + ... + w_n = 0$. **2**

(ii) Show that $(PA_i)^2 = (z - w_i)(\bar{z} - \bar{w_i})$ for *i* = 1, 2, … , *n*. **1**

(iii) Prove that $\sum_{i=1}^{n} (PA_i)^2 = 2n$ **2**

End of paper

2014 Year 12 Mathematics Extension 2 HSC Task 1 Solutions

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
SECTION I 1. $z^3+(i)^3$ B $= (z+i)(z^2-iz-1)$ 2. z^3+1 $=(z+1)(z^2-z+1)$ So $w^3+1=(w+1)(w^2-w+1)$ Now $(w-1)^2$ $= w^2-2w+1$ $= -w$ D 3. $(1+i)^{10} = (\sqrt{2}\operatorname{cis}\frac{\pi}{4})^{10}$ $= 32\operatorname{cis}\frac{5\pi}{2}$ $= 32\operatorname{cis}\frac{\pi}{2}$ $= 32i$ B 4. $\operatorname{Arg}\left(\frac{1}{z}\right) = -\arg z$ $\operatorname{Arg}\left(\frac{i}{z}\right) = \frac{\pi}{2} - \arg z$ D		SECTION II **Question 5** a) (i) $2-3i+(-1-2i)$ $= 1+i$ ii) $z^2 = (2+3i)^2$ $= -5+12i$ iii) $\frac{z}{w} = \frac{2+3i}{-1+2i}\cdot\frac{-1-2i}{-1-2i}$ $= \frac{4-7i}{5}$ $= \frac{4}{5} - \frac{7i}{5}$ b) Let $z^2 = 21-20i$ With $z = a+bi$ So $(a+bi)^2 = 21-20i$ $a^2-b^2 = 21$ $ab = -10$ Also $a^2+b^2 = \|(a+bi)^2\|$ $= \|21-20i\|$ $= 29$ Solving simultaneously, $a=5 \quad b=-2$ $a=-5 \quad b=2$	 ✓ ✓ ✓

$z = \pm(5-2i)$

2014 Year 12 Mathematics Extension 2 HSC Task 1 Solutions

	Comments	Suggested Solution (s)	Comments

(c)(i) $\overrightarrow{BA} = z_1 - z_2$

& $\overrightarrow{BC} = z_3 - z_2$

Rotating $\overrightarrow{BC}$ anticlockwise by 90° gives $\overrightarrow{BA}$.

Hence $z_1 - z_2 = i(z_3 - z_2)$

Squaring both sides gives

$(z_1 - z_2)^2 = -(z_3 - z_2)^2$.

(ii) $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$.

But $\overrightarrow{AD} = \overrightarrow{BC}$.

$\therefore \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC}$

$= z_1 + (z_3 - z_2)$

So D is represented by the complex number $z_1 - z_2 + z_3$. ✓

(d) To show $(1+i)^n + (1-i)^n = 2(\sqrt{2})^n \cos\frac{n\pi}{4}$

LHS $= \left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)^n + \left(\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)\right)^n$

$= 2^{\frac{n}{2}}\left[\cos\frac{n\pi}{4} + \cos\left(-\frac{n\pi}{4}\right) + i\left(\sin\left(\frac{n\pi}{4}\right) + \sin\left(-\frac{n\pi}{4}\right)\right)\right]$

$= 2 \cdot 2^{n/2} \cos\frac{n\pi}{4}$

$= 2(\sqrt{2})^n \cos\frac{n\pi}{4}$

(e) $T_1 = 1$, $T_2 = 5$

$T_{k-1} = 3^{k-1} - 2^{k-1}$

$T_k = 3^k - 2^k$

RTP $T_{k+1} = 5T_k - 6T_{k-1}$

Now

$5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1})$

$= 5\times 3^k - 6\times 3^{k-1} - 5\times 2^k + 6\times 2^{k-1}$

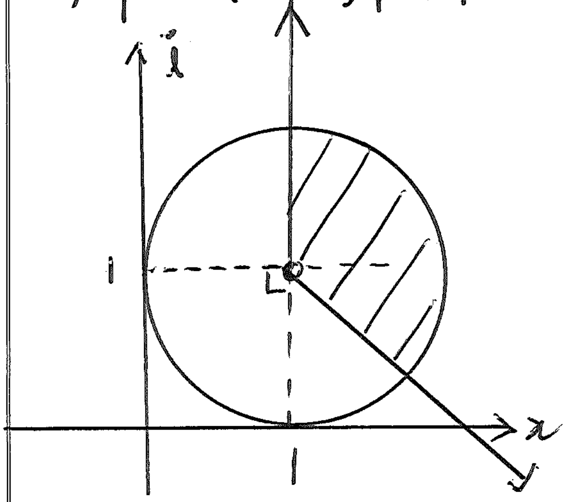
$= 5\times 3^k - 2\times 3^k - 5\times 2^k + 3\times 2^k$

$= 3\times 3^k - 2\times 2^k$

$= 3^{k+1} - 2^{k+1}$

Question 6

a) $|z - (1+i)| \le 1$



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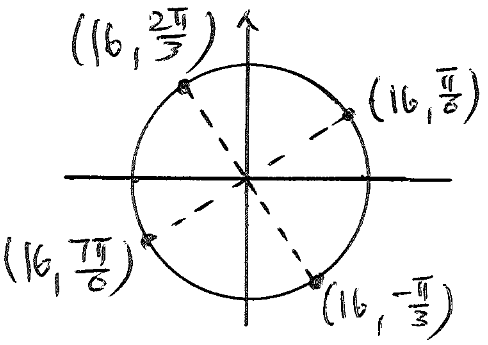
	Comments	Suggested Solution (s)	Comments
b) $Z_K = 16\,\text{cis}\left(\frac{\frac{2\pi}{3}+2k\pi}{4}\right)$ $k=0,1,2,3$ $= 16\,\text{cis}\left(\frac{2\pi+6\pi k}{12}\right)$ $= 16\,\text{cis}\left(\frac{\pi+3k\pi}{6}\right)$ $= 16\,\text{cis}\,(3k+1)\frac{\pi}{6}$ $z_0 = 16\,\text{cis}\frac{\pi}{6}$ $z_1 = 16\,\text{cis}\frac{4\pi}{6} = 16\,\text{cis}\frac{2\pi}{3}$ $z_2 = 16\,\text{cis}\frac{7\pi}{6}$ $z_3 = 16\,\text{cis}\left(\frac{10\pi}{6}\right)$ $= 16\,\text{cis}\frac{5\pi}{3}$ $= 16\,\text{cis}\left(-\frac{\pi}{3}\right)$  		c) Let $P(n)$ be the given proposition $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n} > \ln(x+1)$ $P(1)$ is true since $\frac{1}{1} > \ln(1+1)$ (Note : $e>2 \Rightarrow \ln e > \ln 2$ $\Rightarrow 1 > \ln 2$) Assume $P(k)$ is true for some positive integer k. ie) $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k} > \ln(k+1)$ Prove true for $P(k+1)$ $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}+\frac{1}{k+1}$ $> \ln(k+1)+\frac{1}{k+1}$ (using the assumption) $> \ln(k+1)+\ln\left(1+\frac{1}{k+1}\right)$ (using the inequality given) $= \ln\left((k+1)\left(1+\frac{1}{k+1}\right)\right)$ $= \ln\left((k+1)+1\right)$	

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d) (i) $w^n = 1$ (w is an n^{th} root of unity) Consider $w + w^2 + \cdots + w^n = 0$ $S_n = \dfrac{a(1-r^n)}{1-r}$ $S_n = \dfrac{w(1-w^n)}{1-w}$ but $w^n = 1$ $= \dfrac{w(1-1)}{1-w}$ $= 0$ ii) $\lvert PA_i \rvert = \lvert z - w_i \rvert$ $PA_i^2 = (z - w_i)^2$ $= (z-w_i)\overline{(z-w_i)}$ since $\lvert z\rvert^2 = z \cdot \bar{z}$ $= (z-w_i)(\bar{z}-\bar{w}_i)$		(iii) $\sum_{i=1}^{n} PA_i^2 = \sum_{i=1}^{n} (z-w_i)(\bar{z}-\bar{w}_i)$ $= \sum_{i=1}^{n} (z\bar{z} - z\bar{w}_i - w_i\bar{z} + w_i\bar{w}_i)$ $= \sum_{i=1}^{n} (2 - z\bar{w}_i - w_i\bar{z})$ since $z\bar{z} = \lvert z\rvert^2 = 1$ $w\bar{w} = \lvert w\rvert^2 = 1$ $= 2n - z\sum_{i=1}^{n} \bar{w}_i - \bar{z}\sum_{i=1}^{n} w_i$ $= 2n - z \cdot 0 - \bar{z} \cdot 0$ (by (i)) $= 2n$	