Question 1 (15 Marks)

a) Let $z_1 = 1 - i$ and $z_2 = -1 + i\sqrt{3}$ **4**

i) Find $z_1 z_2$ in the form $a + ib$.

ii) Find $|z_1 z_2|$ and $Arg(z_1 z_2)$. Hence write $z_1 z_2$ in modulus-argument form.

iii) Hence or otherwise show that $\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

b) z is a complex number such that $|z| = 1$. Using an Argand diagram or otherwise show that: **3**

i) $1 \le |z+2| \le 3$

ii) $-\frac{\pi}{6} \le \arg(z+2) \le \frac{\pi}{6}$

c) If $1, \omega_1$ and ω_2 are the cube roots of unity, prove that **4**

i) $\omega_1 = \overline{\omega_2} = \omega_2{}^2$

ii) $\omega_1 + \omega_2 = -1$

iii) $\omega_1 \omega_2 = 1$

d) Let $z = x + iy$ be any non-zero complex number. **4**

i) Express $z + \frac{1}{z}$ in the form $a + ib$

ii) Given that $z + \frac{1}{z} = k$, where k is real show that $y = 0$ or $x^2 + y^2 = 1$

iii) Show also that if $y = 0$ then $|k| \ge 2$ and that if $x^2 + y^2 = 1$ then $|k| \le 2$

Question 2 (15 Marks) ***Start a new page***

a) 7

i) Prove the identity $\cos^3 A - \frac{3}{4}\cos A = \frac{1}{4}\cos 3A$

ii) Show that $x = 2\sqrt{2}\cos A$ satisfies the equation $x^3 - 6x = -2$ provided $\cos 3A = -\frac{1}{2\sqrt{2}}$

★ iii) Find the three roots of the equation $x^3 - 6x + 2 = 0$, leaving your answers to four decimal places.

b) Let $x = \alpha$ be a root of the polynomial $P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$, where $(2+B)^2 \neq 4A^2$ 8

i) Show that α can not be 0, 1 or -1.

ii) Show that $x = \frac{1}{\alpha}$ is a root.

★ iii) Deduce that if α is a multiple root, then its multiplicity is 2 and $4B = 8 + A^2$

Question 3 (15 Marks) ***Start a new page***

a) 3

i) Show that $(x+2)$ is a factor of the polynomial $P(x) = 2x^3 + 19x^2 + 22x - 16$

ii) Find the zeros of the function $P(x) = 2x^3 + 19x^2 + 22x - 16$

b) Let P(x) be a polynomial where $P(x) = x^3 - x^2 - 8x + 12$. 7

i) Factorize P(x), given that the equation P(x) = 0 has roots α, α and β

ii) Sketch the graph of P(x).

iii) Let Q(x) be another polynomial where $Q(x) = P(x)(x + a)$ and a is a constant chosen so that $Q(x) \geq 0$ for all x. Find the value(s) of a.

c) Using part (b) above or otherwise find the area under the curve $y = \frac{x^3 - x^2 - 8x + 13}{x - 2}$ from $x = 3$ to $x = 4$, leaving your answer correct to three significant figures. 4

END OF PAPER

Question 1

a)

i) $z_1 z_2 = \left(\sqrt{3}-1\right) + i\left(\sqrt{3}+1\right)$ ①

ii) $|z_1| = \sqrt{2},\quad |z_2| = 2 \Rightarrow |z_1 z_2| = 2\sqrt{2}$ ①

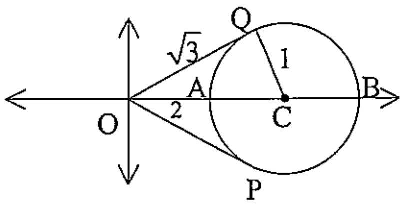
$\arg(z_1) = -\frac{\pi}{4},\quad \arg(z_2) = \frac{2\pi}{3} \Rightarrow \arg(z_1 z_2) = \frac{5\pi}{12}$ ①

$\therefore z_1 z_2 = 2\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

iii) $2\sqrt{2}\cos\frac{5\pi}{12} = \sqrt{3}-1 \Rightarrow \cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ ①

b)

i) $z+2$ lies on a circle with centre $(2,0)$ and radius 1 ①



$1 = OA \le |z+2| \le OB = 3$

ii) $-\frac{\pi}{6} = \angle COQ \le \arg(z+2) \le \angle POQ = \frac{\pi}{6}$ ①

c)

i) wlog let $\omega_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $\omega_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

then $\overline{\omega_2} = \cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

and $\omega_2^{\,2} = \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ ①

ii) 1 , ω_1 , ω_2 are the roots of $z^3 - 1 = 0$ ①

$\therefore$ the sum of the roots $= 1 + \omega_1 + \omega_2 = 0$ ①

iii) $\therefore$ the product of the roots $= \omega_1 \omega_2 = 1$ ①

d)

i) $z + \frac{1}{z} = x + iy + \frac{1}{x+iy} = x + iy + \frac{x-iy}{x^2+y^2} = \left(x + \frac{x}{x^2+y^2}\right) + i\left(y - \frac{y}{x^2+y^2}\right)$ ①

ii) From i) $y - \frac{y}{x^2+y^2} = 0 \Rightarrow y = 0$ or $x^2 + y^2 = 1$ ①

iii) if $y = 0$ then $x + \frac{1}{x} = k \Rightarrow x^2 - kx + 1 = 0$ has real solution $\Rightarrow \Delta \ge 0 \Rightarrow |k| \ge 2$ ①

if $x^2+y^2=1$ then $2x = k$ and since $|x| \le 1$ then $|k| \le 2$ ①

Question 2

a)

i) $\cos 3A + i\sin 3A = (\cos A + i\sin A)^3$ ①

$= \cos^3 A + 3i\cos^2 A\sin A - 3\cos A\sin^2 A - i\sin^3 A$

$\therefore \quad \cos 3A = \cos^3 A - 3\cos A\sin^2 A$

$= \cos^3 A - 3\cos A - 3\cos^3 A$ ①

$\therefore \quad \frac{1}{4}\cos 3A = \cos^3 A - \frac{3}{4}\cos A$ ①

ii) If $x = 2\sqrt{2}\cos A$ satisfies $x^3 - 6x + 2 = 0$ then,

$16\sqrt{2}\cos^3 A - 12\sqrt{2}\cos A = -2$ ①

$\therefore \cos^3 A \frac{3}{4}\cos A = -\frac{1}{8\sqrt{2}}$

$\therefore \cos 3A = -\frac{1}{2\sqrt{2}}$ ①

iii) $\therefore 3A = \pm\cos^{-1}\left(-\frac{1}{2\sqrt{2}}\right) + 2n\pi$ ①

$\therefore A = 0.64405, \quad 2.73845, \quad 4.83284$ ①

$\therefore x \approx 0.64405, \quad 2.73845, \quad 4.83284$ ①

b)

i) $P(0) = 1 \neq 0$

$P(1) = 2A + B + 2 \neq 0$ as $2+B \neq -2A$

$P(-1) = -2A + B + 2 \neq 0$ as $2+B \neq 2A$ ①

ii) $P\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha^4} + \frac{A}{\alpha^3} + \frac{B}{\alpha^2} + \frac{A}{\alpha} + 1 = \frac{1}{\alpha^4}\left(1 + A\alpha + B\alpha^2 + A\alpha^3 + \alpha^4\right) = \frac{1}{\alpha^4}P(\alpha) = 0$ ①

iii) Note that $\alpha \neq \frac{1}{\alpha}$ as $\alpha \neq 1, -1$ ①

$\therefore P(x) = (x-\alpha)^2\left(x - \frac{1}{\alpha}\right)(ax+b)$ ①

$\therefore a = 1$ and $-\alpha b = 1 \Rightarrow b = -\frac{1}{\alpha}$ ①

$\therefore P(x) = (x-\alpha)^2\left(x - \frac{1}{\alpha}\right)^2$ ①

$\therefore P(x) = \left(x^2 - \left(\alpha + \frac{1}{\alpha}\right)x + 1\right)^2$

$\therefore P(x) = x^4 - 2\left(\alpha + \frac{1}{\alpha}\right)x^3 + \left[2 + \left(\alpha + \frac{1}{\alpha}\right)^2\right]x^2 - 2\left(\alpha + \frac{1}{\alpha}\right)x + 1$ ①

$\therefore A = -2\left(\alpha + \frac{1}{\alpha}\right), \quad B = 2 + \left(\alpha + \frac{1}{\alpha}\right)^2$

$\therefore 4B = 8 + A^2$ ①

Question 3

a)

i) $P(-2)=0$ ①

ii)

	2	19	22	-16
-2		-4	-30	16
	2	15	-8	0

①

$\therefore P(x)=(x+2)\left(2x^2+15x-8\right)=(x+2)(2x-1)(x+8)$ ①

$\therefore x=-2,\ \frac{1}{2},\ -8$

b)

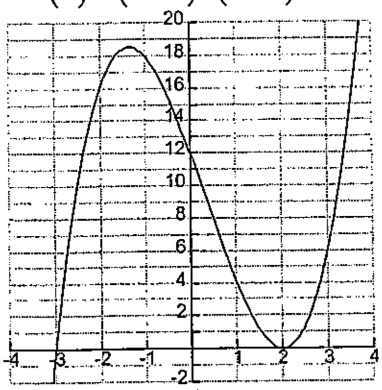
i) Let the roots be $\alpha,\ \alpha,\ \beta$

$2\alpha+\beta=1$ and $\alpha^2+2\alpha\beta=-8$ ①

$\alpha^2+2\alpha-4\alpha^2=-8 \Rightarrow \alpha=2$ or $-\frac{4}{3} \Rightarrow \beta=-3$ or $\frac{11}{3}$ ①

but $\alpha^2\beta=-12 \Rightarrow \alpha=2\ ,\ \beta=-3$

$\therefore P(x)=(x-2)^2(x+3)$ ①



ii) ①

iii) $Q(x)=(x-2)^2(x+3)(x+a)\geq 0\ \ \forall x$

$\therefore x^2+(3+a)x+3a\geq 0$ ①

$\therefore (3+a)^2-12a\leq 0 \Rightarrow (a-3)^2\leq 0 \Rightarrow a=3$ ①

c) $\displaystyle\int_3^4\left(\frac{(x-2)^2(x+3)}{x-2}+\frac{1}{x-2}\right)dx=\left[\frac{x^3}{3}+\frac{x^2}{2}-6x+\ln(x+2)\right]_3^4$

$=\frac{59}{6}+\ln 2\approx 10.5$