QUESTION ONE *(6 Marks)*

MULTIPLE CHOICE: Write the correct alternative on your writing paper.

1. Which of the following gives $\sqrt{3}-i$ in modulus-argument form **1**

(A) $2 \text{ cis } (-\frac{\pi}{6})$

(B) $2 \text{ cis } (-\frac{2\pi}{3})$

(C) $2 \text{ cis } (\frac{\pi}{6})$

(D) $2 \text{ cis } (\frac{2\pi}{3})$

2. Which of the following represent Re(Z) if $Z = \dfrac{4+3i}{1-2i}$ **1**

(A) $\dfrac{-10}{3}$

(B) $\dfrac{-2}{5}$

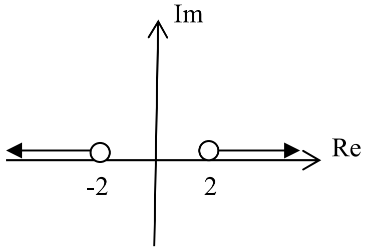
(C) 4

(D) -2

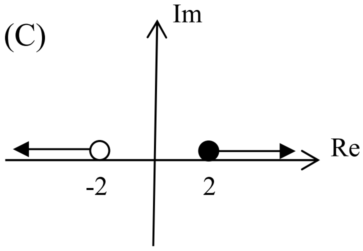
3. Which of the following represent the locus of Z satisfying the condition **1**

$$\arg\left(\frac{Z-2}{Z+2}\right)=0$$

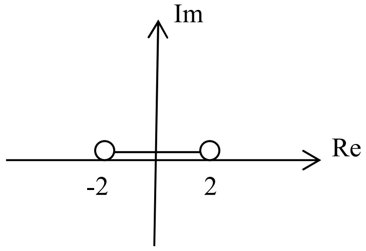
(A)



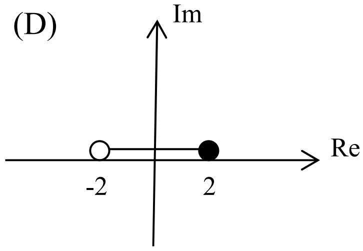
(C)



(B)



(D)



4. Which of the following are the square roots of $-16+30i$? **1**

(A) $\pm(3-5i)$

(B) $\pm(5-3i)$

(C) $\pm(5+3i)$

(D) $\pm(3+5i)$

5. Which is the simplified form of i^{2011}? **1**

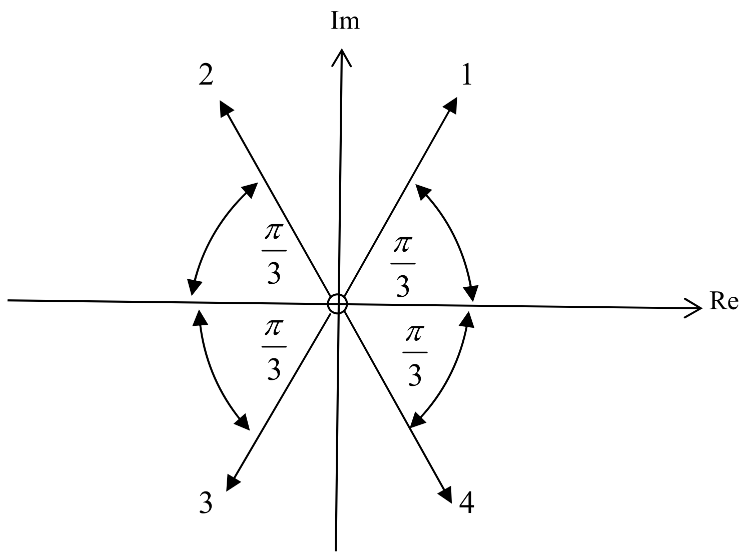
(A) i

(B) -1

(C) $-i$

(D) 1

6. Which ray represents the locus $\arg(-Z)=\dfrac{\pi}{3}$? **1**



(A) 1

(B) 2

(C) 3

(D) 4

QUESTION TWO *(6Marks)* **Start a new page**

(a) Let $z = 4 - i$. **3**

(i) Find z^2 in the form $x + iy$.

(ii) Find $z + 2\overline{z}$ in the form $x + iy$.

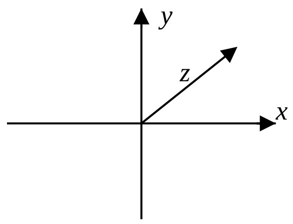
(iii) Find $\frac{i}{z}$ in the form $x + iy$.

(b) Show that $\left(-\sqrt{3} - i\right)^6$ is a real number. **3**

QUESTION THREE *(12 Marks)* **Start a new page**

(a) Let A be the point representing $-1 - i$ and B be the point representing $3 + i$ in the Argand plane. Draw a sketch of the locus of the points z, so that $|z - A| \le |z - B|$. **2**

(b)



The diagram above shows a vector representing the complex number z. Copy the diagram and on it show vectors representing the complex numbers iz and z-iz. **3**

(c) The points O, I, Z and P on the Argand diagram represent the complex numbers 0, 1, z and $z + 1$ respectively, where $z = \cos\theta + i\sin\theta$ is any complex number of modulus 1 and $0 < \theta < \pi$. **3**

(i) Explain why $OIZP$ is a rhombus.

(ii) Show that $\frac{z-1}{z+1}$ is purely imaginary.

(c) (i) Express $(-1 + \sqrt{3}i)(1 + i)$ in the form $a + ib$ **4**

(ii) Hence, or otherwise, find the exact value of $\cos\frac{11\pi}{12}$

QUESTION FOUR *(16Marks)* **Start a new page**

(a) (i) Write down the general solution of $\tan 4\theta = 1$. **9**

(ii) Use De Moivre's Theorm to express $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

(iii) Hence show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

(iv) Hence or otherwise, find the roots of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$.

(v) Hence show that $\tan\dfrac{\pi}{16} + \tan\dfrac{5\pi}{16} = \tan\dfrac{3\pi}{16} + \tan\dfrac{7\pi}{16} - 4$.

(b) (i) Find all solutions to the equation $z^6 = 1$ in the form $x + iy$. **7**

(ii) If ω is the non real solution to the equation $z^6 = 1$, show that $\omega^4 + \omega^2 = -1$.

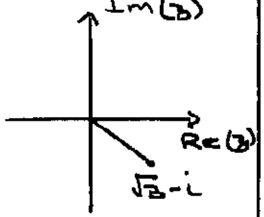
(iii) By choosing on particular value of ω, explain with the aid of a diagram why $\omega^4 + \omega^2 = -1$.

END OF PAPER

Year 12 Extension 2 Assessment 1 2012

Question One

1. $z = \sqrt{3} - i$



$|z| = \sqrt{(\sqrt{3})^2 + 1^2}$

$= 2$

$\arg(z) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{-\pi}{6}$

$\therefore \sqrt{3} - i = 2\,\text{cis}\left(\frac{-\pi}{6}\right)$ (A)

2. $z = \frac{4+3i}{1-2i}$

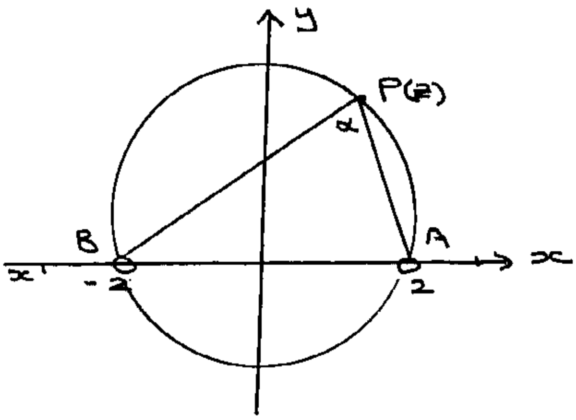
$= \frac{(4+3i)}{(1-2i)} \times \frac{(1+2i)}{(1+2i)}$

$= \frac{4+11i-6}{1+4}$

$= -\frac{2}{5} + \frac{11}{5} i$

$\therefore \text{Re}(z) = -\frac{2}{5}$ (B)

3.

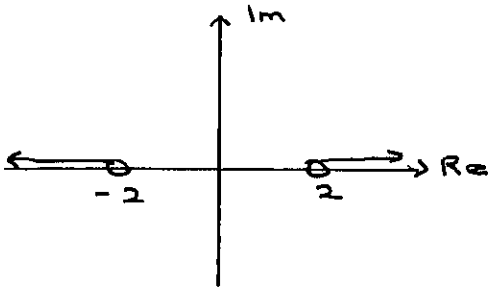


$\arg\left(\frac{z-2}{z+2}\right) = \arg(z-2) - \arg(z+2)$

$= 0$

If $\alpha = 0$, $P\hat{A}x = P\hat{B}x$

$\therefore$ locus of P is the rays Ax, and Bx'



(A)

4. Let $x + iy = \sqrt{-16+30i}$

squaring both sides

$(x+iy)^2 = -16 + 30i$

$x^2 - y^2 + 2xyi = -16 + 30i$

equating real and imaginary parts

$x^2 - y^2 = -16$

$2xy = 30$

$x = \pm 3$

$y = \pm 5$

$\therefore$ The 2 square roots are $3+5i$ and $-3-5i$.

that is $\pm(3+5i)$ (D)

5. $\frac{2011}{4} = 502\frac{3}{4}$

$\therefore i^{2011} = i^3$

$= -i$ (C)

6. (C)

Question Two

a) (i) Let $z = 4 - i$

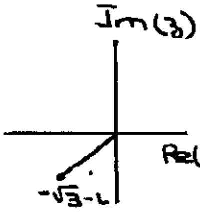
$$z^2 = (4-i)^2$$
$$= 16 - 8i + i^2$$
$$= 15 - 8i$$

(ii) $z + 2\bar{z} = 4 - i + 2(4+i)$
$$= 4 - i + 8 + 2i$$
$$= 12 + i$$

(iii) $\frac{i}{z} = \frac{i}{4-i}$
$$= \frac{i}{(4-i)} \times \frac{(4+i)}{(4+i)}$$
$$= \frac{-1+4i}{17}$$
$$= -\frac{1}{17} + \frac{4}{17}i$$

b) Let $z = -\sqrt{3} - i$

$$|z| = \sqrt{(\sqrt{3})^2 + 1^2}$$
$$= 2$$



$$\arg(z) = \tan^{-1}\frac{1}{\sqrt{3}}$$
$$= -\frac{5\pi}{6}$$

$\therefore -\sqrt{3} - i = 2\,\text{cis}\left(\frac{-5\pi}{6}\right)$

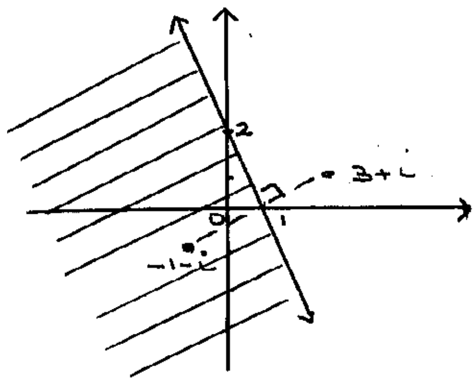
Using De Moivre's Theorem

$$(-\sqrt{3}-i)^6 = \left[2\,\text{cis}\left(\frac{-5\pi}{6}\right)\right]^6$$
$$= 2^6\,\text{cis}(-\pi)$$
$$= -64$$

$\therefore (-\sqrt{3}-i)^6$ is a real number

Question Three

a)



$(-1,-1)$ $(3,1)$

$$\text{midpoint} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$
$$= \left(\frac{-1+3}{2}, \frac{-1+1}{2}\right)$$
$$= (1, 0)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{1+1}{3+1}$$
$$= \frac{1}{2}$$

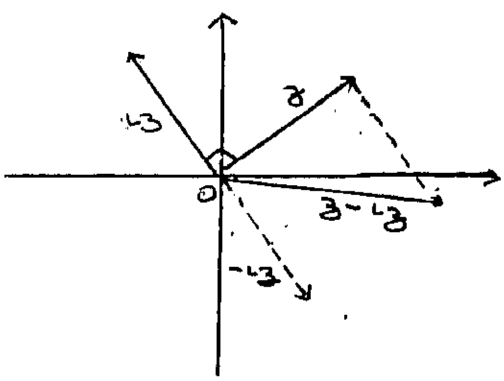
$\therefore$ Required gradient $= -2$ as $m_1 m_2 = -1$ for perpendicular lines

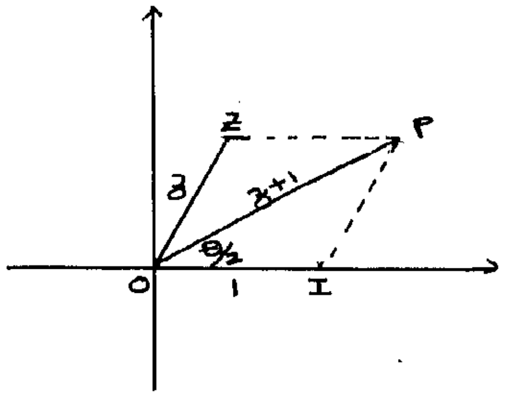
$$y - y_1 = m(x - x_1)$$
$$y - 0 = -2(x - 1)$$
$$y = -2x + 2$$

b) Let $z = x + iy$

$$iz = -y + ix$$

$\therefore z$ is rotated $90°$ in an anticlockwise direction





(i) $z = \cos\theta + i\sin\theta$

$|z| = 1$

OIPZ is a parallelogram

$OI = OZ$

$= 1$

$\therefore$ as the adjacent sides are equal OIZP is a rhombus

(ii) $OP \perp ZI$ as the diagonals bisect at right angles

OP represents $z+1$

OI represents $z-1$

$\therefore \; z - 1 = ki(z+1)$

$\therefore \; \dfrac{z-1}{z+1}$ is purely imaginary

c) $(-1+\sqrt{3}i)(1+i) = -1 - i + \sqrt{3}i - \sqrt{3}$

$= -(1+\sqrt{3}) + (\sqrt{3}-1)i$

Let $z = -1 + \sqrt{3}i$

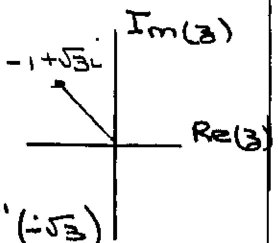
$|z| = \sqrt{1^2 + (\sqrt{3})^2}$

$= 2$

$\arg z = \pi - \tan^{-1}(\div\sqrt{3})$

$= \frac{2\pi}{3}$

$\therefore \; -1 + \sqrt{3}i = 2\,\text{cis}\left(\frac{2\pi}{3}\right)$



$\omega = (1+i)$

$|\omega| = \sqrt{1^2+1^2}$

$= \sqrt{2}$

$\arg(\omega) = \tan^{-1} 1$

$= \frac{\pi}{4}$

$\therefore \; 1+i = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

$\therefore \; (-1+\sqrt{3}i)(1+i) = 2\,\text{cis}\left(\frac{2\pi}{3}\right)\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

$= 2\sqrt{2}\,\text{cis}\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$

$= 2\sqrt{2}\,\text{cis}\left(\frac{11\pi}{12}\right)$

Equating real parts

$2\sqrt{2}\cos\frac{11\pi}{12} = -(1+\sqrt{3})$

$\cos\frac{11\pi}{12} = \dfrac{-(1+\sqrt{3})}{2\sqrt{2}}$

__Question Four__

a) (i) $\tan 4\theta = 1$

$\tan\frac{\pi}{4} = 1$

$\therefore \; 4\theta = n\pi + \frac{\pi}{4}$

$\theta = \frac{n\pi}{4} + \frac{\pi}{16} \quad n \in \mathbb{Z}$

ii) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ by De Moivre's Theorem

$$L.H.S = (\cos\theta + i\sin\theta)^4$$

$$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$

$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

Equating real parts

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \quad \text{——— ①}$$

Equating imaginary parts

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \quad \text{——— ②}$$

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$$

$$= \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

(iii) Now L.H.S = 1 when $\tan 4\theta = 1$

Let $x = \tan\theta$

$$\therefore \quad 1 = \frac{4x - 4x^3}{1 - x^2 + x^4}$$

$$x^4 - x^2 + 1 = 4x - 4x^3$$

$$x^4 + 4x^3 - x^2 - 4x + 1 = 0 \quad \text{as required}$$

$$\therefore \tan 4\theta = 1$$

$$4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$ NB from (a)

$$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$$ $\theta = \frac{n\pi}{4} + \frac{\pi}{16}$ for $n = 0, 1, 2, 3$

$$\therefore x = \tan\theta$$

$$x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$$

$$\therefore x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, -\tan\frac{7\pi}{16}, -\tan\frac{3\pi}{16}$$

as tan θ is negative in the second quadrant

$\therefore$

(iv) the sum of the roots $= \frac{-b}{a}$

$= -4$

$\therefore \tan\frac{\pi}{16} + \tan\frac{5\pi}{16} - \tan\frac{7\pi}{16} - \tan\frac{3\pi}{16} = -4$

$\therefore \tan\frac{\pi}{16} + \tan\frac{5\pi}{16} = \tan\frac{3\pi}{16} + \tan\frac{7\pi}{16} - 4$

as required.

b) (i) $z^6 = 1$

$z^6 = \text{cis}(2k\pi)\ k \in \mathbb{Z}$

$z = \text{cis}\left(\frac{2k\pi}{6}\right)$ by De Moivre's Theorem

$= \text{cis}\left(\frac{k\pi}{3}\right)$

For $k = 0$

$z = \text{cis}\, 0$

$z = 1$

$k = 1$

$z = \text{cis}\left(\frac{\pi}{3}\right)$

$= \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

$\therefore z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$k = 2$

$z = \text{cis}\left(\frac{2\pi}{3}\right)$

$= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

$z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$k = 3$

$z = \text{cis}\,\pi$

$= \cos\pi + i\sin\pi$

$= -1$

$k = -1$

$z = \text{cis}\left(-\frac{\pi}{3}\right)$

$= \cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$k = -2$

$z = \text{cis}\left(-\frac{2\pi}{3}\right)$

$z = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\therefore$ The solutions are $\pm 1,\ \frac{1}{2} \pm \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

(ii) $z^6 - 1 = 0$

$(z^3 - 1)(z^3 + 1) = 0$

$(z-1)(z^2+z+1)(z+1)(z^2-z+1) = 0$

The 2 real roots are given by $(z-1)(z+1)$

The 4 non real roots are given by $(z^2+z+1)(z^2-z+1)$

let ω be the complex roots

$(\omega^2+\omega+1)(\omega^2-\omega+1) = 0$

$\omega^4 - \omega^3 + \omega^2 + \omega^3 - \omega^2 + \omega + \omega^2 - \omega + 1 = 0$

$\omega^4 + \omega^2 + 1 = 0$

$\therefore \omega^4 + \omega^2 = -1$ as required

(iii) From (i) Let $\omega = cis(\frac{\pi}{3})$

$\omega^4 = cis(\frac{4\pi}{3})$ by De Moivres Theorem

$\omega^2 = cis(\frac{2\pi}{3})$

$$cis\frac{4\pi}{3} + cis\frac{2\pi}{3} = \cos(\frac{4\pi}{3}) + i\sin(\frac{4\pi}{3}) + \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i + -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$= -1$$