Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2009

ASSESSMENT TASK 1

Mathematics

Extension 2

General Instructions

- Working time – 60 minutes
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Barrett
- O Mr Fletcher
- O Mr Weiss

Student Number:________________________

(To be used by the exam markers only.)

Question No	1	2	3	4	Total	Total
Mark	$\overline{20}$	$\overline{15}$	$\overline{9}$	$\overline{8}$	$\overline{52}$	$\overline{100}$

Question 1 (20 marks)

(a) If $z = 1 + 2i$ and $w = 3 - i$, find in the form $a + ib$

(i) $z\overline{w}$ **2**

(ii) $\dfrac{z}{w}$ **2**

(b) (i) Express $-1 + i$ in modulus-argument form **2**

(ii) **Hence** evaluate $(-1 + i)^{11}$, giving your answer in the form $a + ib$. **2**

(iii) What is the least positive integer for which $(-1 + i)^n$ is a positive integer? **1**

(c) Sketch the following loci on an Argand diagram:

(i) $|z + 3 - 4i| = 5$ **2**

(ii) $|z| \le |z - (2 + 2i)|$ and $-\pi \le \operatorname{Arg} z \le \dfrac{\pi}{4}$ **3**

(d) (i) Show that $\sqrt{8 - 6i} = \pm(3 - i)$ **3**

(ii) Hence solve $6z^2 - 4iz + (i - 2) = 0$, giving answers in the form $a + ib$. **3**

Question 2 (15 marks)

(a) Consider the polynomial $P(z) = z^4 - 4z^3 + 6z^2 - 4z + 5$.

(i) Show that $z + i$ is a factor of $P(z)$. **2**

(ii) Hence state why $z - i$ is also a factor of $P(z)$. **1**

(iii) Form the monic quadratic equation in z whose roots are $\pm i$. **1**

(iv) Hence factorise $P(z)$ fully over the complex numbers. **2**

(b) Given that the polynomial $P(z) = z^3 + 2z^2 - z + 1$ has zeros α, β and γ, find the polynomial function whose roots are

(i) $\alpha - 1, \beta - 1$ and $\gamma - 1$ **2**

(ii) $(\alpha - 1)^2, (\beta - 1)^2$ and $(\gamma - 1)^2$ **2**

hence (iii) find the value of $\dfrac{1}{(\alpha - 1)^2} + \dfrac{1}{(\beta - 1)^2} + \dfrac{1}{(\gamma - 1)^2}$. **2**

(c) Using the polynomial in part (b), form the polynomial which has roots $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$ **3**

Question 3 (9 marks)

(a) When a polynomial is divided by $x - 2$ and $x - 3$, the respective remainders are 4 and 9. Find the remainder when the polynomial is divided by $x^2 - 5x + 6$. **3**

(b) (i) If the polynomial $P(x)$ has a zero α of multiplicity n, prove that $P'(x)$ has the same zero with multiplicity $n - 1$. **3**

(ii) If $ax^3 + bx^2 + d = 0$ (where $d \neq 0$) has a double root, show that **3**

$$27a^2d + 4b^3 = 0.$$

Question 4 (8 marks)

(a) (i) If $z = \cos\theta + i\sin\theta$, use de Moivre's theorem to show that **2**

$$z^n + z^{-n} = 2\cos n\theta.$$

(ii) Hence solve $2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$. **3**

(b) If $|z - \omega| = |z + \omega|$, find the difference between $\operatorname{Arg} z$ and $\operatorname{Arg} \omega$. **3**

Show all reasoning.

Q1) a) i/ $z\bar{w} = (1+2i)(3+i)$ 1

$= 1+7i$ 1

ii/ $\dfrac{z}{w} = \dfrac{1+2i}{3-i} \times \dfrac{3+i}{3+i}$ 1

$= \dfrac{1+7i}{10}$

$= \dfrac{1}{10} + \dfrac{7i}{10}$ 1

b) i/ $-1+i = \overset{1}{\sqrt{2}} \text{ cis } \overset{1}{\dfrac{3\pi}{4}}$

- arg
- mod
- form

} 1 off each error

ii/ $(-1+i)^{11} = (\sqrt{2})^{11} \text{ cis } \dfrac{33\pi}{4}$ 1

$= 32\sqrt{2} \text{ cis } \dfrac{\pi}{4}$

$= 32\sqrt{2}\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)$

$= 32 + 32i$ 1

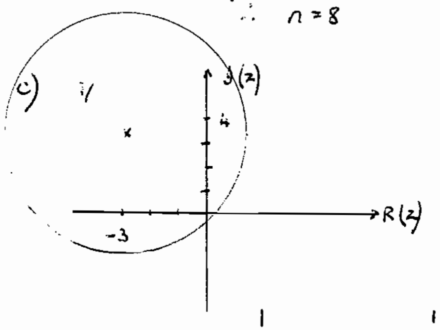
iii/ $n \cdot \dfrac{3\pi}{4} = 2k\pi$, k integer

$n = \dfrac{8k}{3}$

first integer n when $k=3$

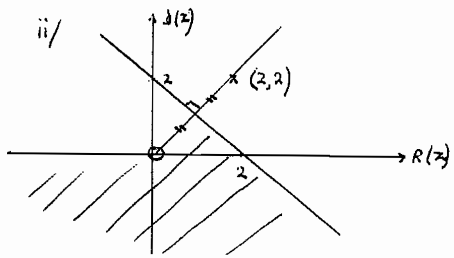
$\therefore n = 8$ 1

c) i/



circle centre $\overset{1}{(-3,4)}$ radius $\overset{1}{5}$

ii/



- region 1 $y < 2-x$
- region 2 $-\pi \le \arg z \le \frac{\pi}{4}$
- correct region shaded
- open circle

1 off per error

d) i/ $\sqrt{8-6i} = a+ib$

$\therefore a^2 - b^2 = 8$

$ab = -3$ 1

$a = \dfrac{-3}{b}$

$\therefore \dfrac{9}{b^2} - b^2 = 8$

$b^4 + 8b^2 - 9 = 0$

$(b^2+9)(b^2-1) = 0$ 1

$\therefore b = \pm 1$ only

when $b = 1$, $a = -3$; $b = -1$, $a = 3$ 1

$\therefore \sqrt{8-6i} = \pm(3-i)$

ii/ $6z^2 - 4iz + (i-2) = 0$

$\therefore z = \dfrac{4i \pm \sqrt{16i^2 - 24(i-2)}}{12}$ 1

$= \dfrac{4i \pm \sqrt{32-24i}}{12}$

$= \dfrac{4i \pm 2\sqrt{8-6i}}{12}$

$= \dfrac{2i \pm (3-i)}{6}$ 1

$= \dfrac{6i-6}{6}$ or $\dfrac{i+3}{6}$

$= \dfrac{i-1}{2}$ or $\dfrac{i+3}{6}$ 1

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Q (2a) i) $P(-i) = (-i)^4 - 4\times -i^3 + 6\times -i^2 - 4\times -i + 5$ (1)

$= 1 - 4i - 6 + 4i + 5$

$= 0$ (1)

$\boxed{2}$

ii) The coefficients of $P(z)$ are real. $\therefore$ complex roots come in conjugate pairs $\therefore z - i$ is a factor

$\boxed{1}$

iii) $z^2 + 1 = 0$

$\boxed{1}$

iv) $P(z) = (z^2+1)Q(x)$

$$\begin{array}{r} z^2 - 4z + 5 \\ z^2+1 \overline{) z^4 - 4z^3 + 6z^2 - 4z + 5} \\ z^4 \quad\quad + z^2 \\ \hline -4z^3 + 5z^2 - 4z \\ -4z^3 \quad\quad - 4z \\ \hline 5z^2 + 5 \\ 5z^2 + 5 \\ \hline \end{array}$$

$\boxed{2}$

$\therefore P(z) = (z^2+1)(z^2-4z+5)$ (1)

$(z-(2+i))(z-(2-i))$ (1)

$z = \frac{4 \pm \sqrt{16-20}}{2}$

$= \frac{4 \pm 2i}{2}$

$= 2 \pm i$

b) i) $P(z+1) = 0$

$\Rightarrow (z+1)^3 + 2(z+1)^2 - (z+1) + 1 = 0$

$z^3 + 3z^2 + 3z + 1 + 2z^2 + 4z + 2 - 2 - 1 + 1 = 0$

$\boxed{2}$

$Q(z) = z^3 + 5z^2 + 6z + 3$

ii) $Q(\sqrt{z})$

$= (\sqrt{z})^3 + 5(\sqrt{z})^2 + 6\sqrt{z} + 3 = 0$

$\boxed{2}$

$= z\sqrt{z} + 5z + 6\sqrt{z} + 3 = 0$

$\sqrt{z}(z+6) = -5z - 3$

$z(z+6)^2 = (5z+3)^2$

$z(z^2+12z+36) = 25z^2 + 30z + 9$

$R(z) = z^3 - 13z^2 + 6z - 9 = 0$

b) iii) $R\left(\frac{1}{z}\right) = 0$

$$= \left(\frac{1}{z}\right)^3 - 13\frac{1}{z^2} + \frac{20}{z} - 9 = 0$$

$$= \frac{1}{z^3} - \frac{13}{z^2} + \frac{20}{z} - 9 = 0$$

$$1 - 13z + 20z^2 - 9z^3 = 0 \qquad \boxed{2}$$

Or $9z^3 - 20z^2 + 13z + 1$

c) $$P(x) = z^3 - (\alpha\beta + \beta\gamma + \alpha\gamma)z^2 + (\alpha\beta^2\gamma + \alpha^2\beta\gamma + \alpha\beta\gamma^2)z + (\alpha\beta\gamma)^2$$

$$= z^3 - (\qquad 11 \qquad)z^2 + \alpha\beta\gamma(\beta + \alpha + \gamma)z + (\alpha\beta\gamma)^2$$

$$= z^3 + 2z^2 + 2z - 1 = 0$$

$\alpha\beta + \beta\gamma + \gamma\alpha = -2$

$\alpha\beta\gamma(\alpha + \beta + \gamma) = 2 \qquad \boxed{3}$

$\alpha\beta\gamma = 1.$

3 (a) $P(x) = (x-2)(x-3)\,Q(x) + ax+b$ 1

$P(2)=4 \Rightarrow \quad 4 = 2a+b$

$P(3)=9 \Rightarrow \quad 9 = 3a+b$ 1

Solve simultaneously $\Rightarrow$ $a=5$, $b=-6$

$\therefore$ remainder $= 5x-6$ 1.

(b) (i) Let $P(x) = (x-\alpha)^n Q(x)$ where $Q(\alpha)\neq 0$ 1

then $P'(x) = (x-\alpha)^n . Q'(x) + n(x-\alpha)^{n-1} Q(x)$ 1

$= (x-\alpha)^{n-1}\left[(x-\alpha)\,Q'(x) + n\,Q(x)\right]$

$Q(\alpha)\neq 0$

$\therefore P'(x)$ has α as a zero of multiplicity $n-1$

(ii) Let $P(x) = ax^3+bx^2+d$

then $P'(x) = 3ax^2+2bx$

$P'(x)=0 \Rightarrow x(3ax+2b)=0$

$x\neq 0$ because $d\neq 0$

$\therefore x = \dfrac{-2b}{3a}$ 1

sub. in $P(x) \Rightarrow$ $a\left(-\dfrac{2b}{3a}\right)^3 + b\left(-\dfrac{2b}{3a}\right)^2 + d = 0$

$\dfrac{-8b^3}{27a^2} + \dfrac{4b^3}{9a^2} + d = 0$

$\dfrac{-8b^3+12b^3}{27a^2} + d = 0$

$\dfrac{4b^3}{27a^2} + d = 0$

$\therefore 27a^2d + 4b^3 = 0$ 1

4 (a) (i) $z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$ (cos is an even function, sin is an odd function)

$\therefore z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ 1

$= 2\cos n\theta$ 1

(ii) $2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$

$z \neq 0$ $\therefore$ divide both sides by z^2

$\Rightarrow 2(z^2 + z^{-2}) + 3(z + z^{-1}) + 5 = 0$

i.e. $4\cos 2\theta + 6\cos\theta + 5 = 0$ 1

i.e. $4(2\cos^2\theta - 1) + 6\cos\theta + 5 = 0$

i.e. $8\cos^2\theta + 6\cos\theta + 1 = 0$

$(4\cos\theta + 1)(2\cos\theta + 1) = 0$

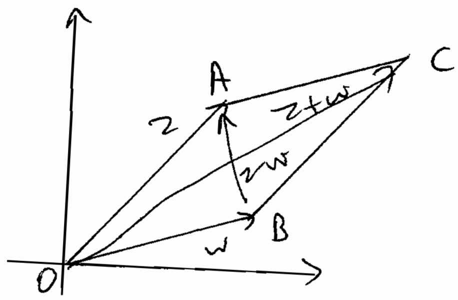
$\therefore \cos\theta = -\frac{1}{4}$ or $\cos\theta = -\frac{1}{2}$ 1

$\cos\theta = -\frac{1}{4} \Rightarrow \sin\theta = \pm\frac{\sqrt{15}}{4}$

$\cos\theta = -\frac{1}{2} \Rightarrow \sin\theta = \pm\frac{\sqrt{3}}{2}$

$\therefore z = -\frac{1}{4} \pm \frac{\sqrt{15}}{4}i$; $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ 1

(b)



Let OA represent z and OB represent w

then $z+w$ and $z-w$ are diagonals of parallelogram OACB

$|z+w| = |z-w| \Rightarrow$ diagonals are equal

i.e. OACB is a rectangle

i.e. $\angle AOB = \frac{\pi}{2}$

But $\angle AOB = \arg z - \arg w$

i.e. $\arg z - \arg w = \frac{\pi}{2}$