Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL
2010
ASSESSMENT TASK 1

Mathematics
Extension 2

General Instructions

- Working time – 50 minutes
- Write in the booklet provided on both sides of the paper
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Barrett
- O Mr Trenwith
- O Mr Weiss

Student Name:__________________________

(To be used by the exam markers only.)

Question No	1	2	3	4	Total	Total
Mark	$\frac{}{9}$	$\frac{}{13}$	$\frac{}{15}$	$\frac{}{12}$	$\frac{}{49}$	$\frac{}{100}$

Question 1 (Start a new page)

a) If P represents the complex number z, sketch the locus of P if

$\text{Im}(z) > 1$ and $0 \leq \arg z \leq \frac{\pi}{4}$ **2**

b) Given $w = \frac{2-3i}{1+i}$. Find: **3**

i) $|w|$

ii) $\bar{w}$

iii) $w + \bar{w}$

c) Given $z = x + iy$, prove that $z + \bar{z} = 2\text{Re}(z)$ **2**

d) Sketch the locus of $\arg\left(\frac{z-1}{z+2i}\right) = \frac{\pi}{3}$ **2**

Question 2 (Start a new page)

a) Prove that if u and v are real the following is purely imaginary **2**

$$\frac{u+iv}{u-iv} - \frac{u-iv}{u+iv}$$

b) Find the modulus and argument of $z = \left(\frac{1+i}{1-i}\right)^2$ and express the answer in

modulus-argument form **3**

c) i) Find (a,b) such that $(3+2i)(a+ib) = 7-5i$ **3**

ii) Find $\sqrt{5-12i}$ **2**

d) If $z_1 = 2 + 5i$ **3**

$z_2 = 3+4i$

express z_1, z_2, and $z_1.z_2$ in modulus-argument form.

Examination continues on next page …

Question 3 (Start a new page)

a) Find the real roots of $x^4 + 2x^3 - 12x^2 + 14x - 5 = 0$, given that it has a triple root. **3**

b) Given that $2 - 3i$ is a zero of $P(x) = x^3+x^2-7x+65$, find the other zeros. **3**

c) Solve $x^3-3x^2-6x+8=0$ given that the roots are in geometric progression. **3**

d) If w is a non-real cube root of unity (i.e. $w^3 = 1$)

i) Prove $\frac{1}{1+w} + \frac{1}{1+w^2} = 1$ **3**

ii) Show $\frac{1+w^n+w^{2n}}{3} = 1$ or 0. According to whether n is or is not a multiple of 3. **3**

Examination continues on next page ...

Question 4 (Start a new page)

a) Let $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

i) Find z^6. **1**

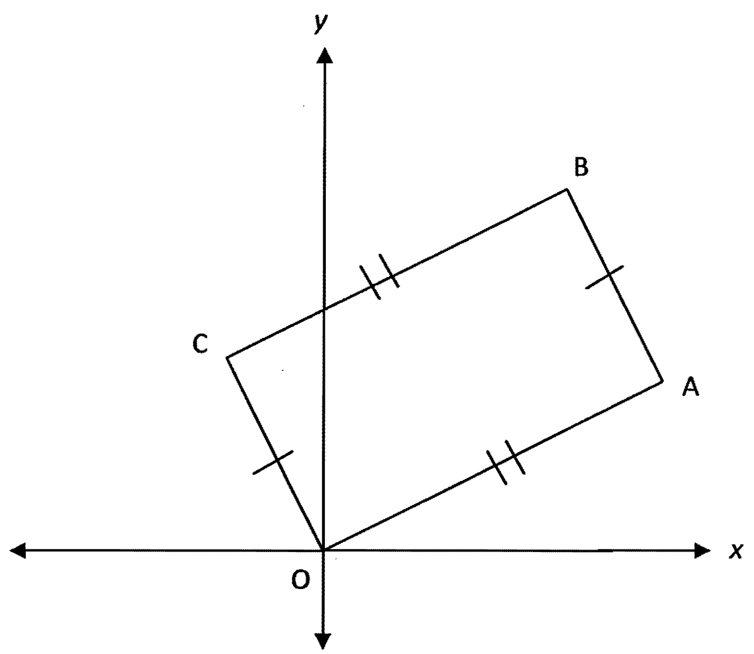
ii) Plot on an Argand diagram, all complex numbers that are solutions of $z^6 = -1$ **2**

iii) Hence or otherwise, find the roots of $z^4 + z^2 + 1 = 0$ **3**

b) Sketch the region on the Argand diagram that satisfies the inequality

$$z\bar{z} + 2(z + \bar{z}) \leq 0$$ **3**

c) In the Argand diagram $OABC$ is a rectangle where $OC = 2OA$. The vertex A corresponds to the complex number w

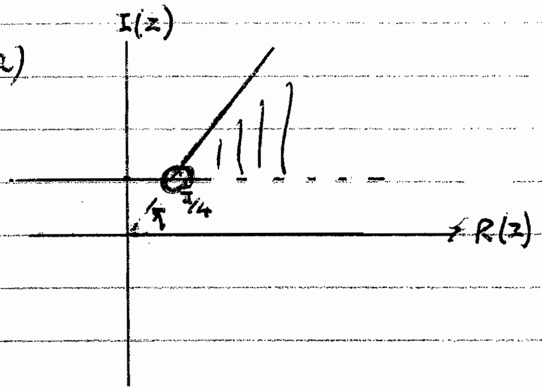


i) What is the complex number that corresponds to the vertex C. **1**

ii) What complex number corresponds to the point of intersection D of the diagonals OB and AC. **2**

End of examination.

Q1) a)



2

1 off for each thing missing
No open circle loose 1 mark

b) i/ $W = \frac{2-3i}{1+i} \times \frac{1-i}{1-i}$

$= \frac{-1-5i}{2}$ 1

ii/ $|W| = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5}{2}\right)^2}$

$= \frac{1}{2}\sqrt{26} = \sqrt{\frac{13}{2}}$ 1

iii/ $\bar{W} = \frac{-1}{2} + \frac{5i}{2}$

$W+\bar{W} = \frac{-1}{2} - \frac{5i}{2} + \frac{-1}{2} + \frac{5i}{2}$ 1

$= -1$

c)

$z = x+iy$

$z+\bar{z} = x+iy+x-iy$

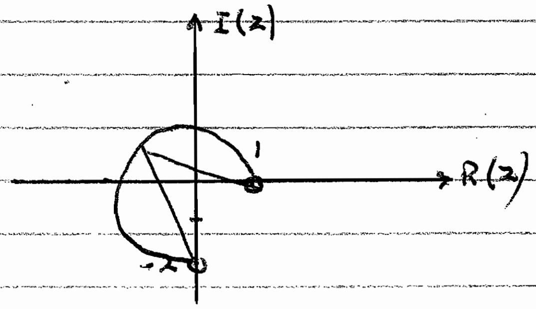
$= 2x$

Now $R(z) = x$

$2R(z) = 2x$

$\therefore 2R(z) = z+\bar{z}$ 2

d)



2

Q2) a) $\dfrac{u+iv}{u-iv} - \dfrac{u-iv}{u+iv}$

$= \dfrac{(u+iv)^2 - (u-iv)^2}{u^2+v^2}$

$= \dfrac{u^2+2iuv-v^2-u^2+2iuv+v^2}{u^2+v^2}$ 1

$= \dfrac{4iuv}{u^2+v^2}$

$= \left(\dfrac{4uv}{u^2+v^2}\right) i$ 1

hence expression is purely imaginary

b) $z = \left(\dfrac{1+i}{1-i}\right)^2$

$= \dfrac{1+2i-1}{1-2i-1}$

$= -1$ 1

$|z| = 1$ 1

$\arg z = \pi$ 1

$\therefore z = 1 \text{ cis } \pi$

c) i/ $(3+2i)(a+ib) = 7-5i$

$\left.\begin{aligned} 3a - 2b &= 7 \\ 3b + 2a &= -5 \end{aligned}\right\}$ 1

$6a - 4b = 14$

$6a + 9b = -15$

$13b = -29$

$b = -\dfrac{29}{13}$ 1

$\therefore a = \dfrac{11}{13}$ 1

ii/ $x+iy = \sqrt{5-12i}$

$x^2-y^2 = 5 \; ; \quad 2xy = -12$

$x^4 - 5x^2 - 36 = 0$

$(x^2-9)(x^2+4) = 0$

$\therefore \sqrt{5-12i} = \pm(3-2i)$ 1 each

accept $\pm\sqrt{13} \text{ cis}\left(\dfrac{\tan^{-1}\left(\frac{-12}{5}\right)}{2}\right)$

$= \pm\sqrt{13} \text{ cis}(-33^\circ)$

d) $z_1 = 2+5i$

$\left.\begin{aligned} |z_1| &= \sqrt{29} \\ \arg z_1 &= 68^\circ 12' \text{ or } 1.19 \text{ or } \tan^{-1}\tfrac{5}{2} \end{aligned}\right\}$ 1

$z_2 = 3+4i$

$\left.\begin{aligned} |z_2| &= 5 \\ \arg z_2 &= 53^\circ 8' \text{ or } 0.93 \text{ or } \tan^{-1}\tfrac{4}{3} \end{aligned}\right\}$ 1

$|z_1 z_2| = |(2+5i)(3+4i)|$

$= |6-20+15i+8i|$

$= |-14+23i|$

$\left.\begin{aligned} |z_1 z_2| &= \sqrt{725} \\ &= 5\sqrt{29} \\ \arg z_1 z_2 &= \tan^{-1}\left(\tfrac{-23}{14}\right) \end{aligned}\right\}$ 1

$= 121^\circ 20'$ or 2.12 or $\tan^{-1}\frac{5}{2} + \tan^{-1}\frac{4}{3}$

Q3) a) $P(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$

$P'(x) = 4x^3 + 6x^2 - 24x + 14$

$P''(x) = 12x^2 + 12x - 24$

$P''(x) = 0$

$12(x+2)(x-1) = 0$

$\therefore x = +1$ or -2 ✓

$P(1) = 1 + 2 - 12 + 14 - 5$ ✓ method for next root.

$= 0$

$\therefore P(x) = (x-1)^3(x+5)$

Roots are 1 (triple) and 5 ✓

b) If $2-3i$ is a root

$2+3i$ is also a root ✓

Now $(2-3i)(2+3i) = 4+9 = 13$ ✓

$\therefore$ other zeros $= \frac{-65}{13} = -5$ ✓

$\therefore$ Zeros are $-5, 2+3i, 2-3i$

c) $\frac{a}{r} \times a \times ar = a^3 = -8$

$\therefore a = -2$ ✓

$\frac{a}{r} + a + ar = 3$

$\frac{-2}{r} - 2 - 2r = 3$

$\frac{-2}{r} - 2r = 5$

$\therefore 2r^2 + 5r + 2 = 0$

$(2r+1)(r+2) = 0$

$\therefore r = \frac{-1}{2}$ or -2

$\therefore a = -2 \quad r = -2 \Rightarrow 1, -2, 4$ ✓✓

$a = -2 \quad r = \frac{1}{2} \qquad 4, -2, 1$ } Roots are $1, -2, 4$.

Marking for c: Answers only for this with wrong method 1 mark only.

✓ for getting the 1st root with series correctly.

✓✓ for the other 2 roots obtained with a correct method.

d) i/ $\frac{1}{1+\omega} + \frac{1}{1+\omega^2} = 1$

$\text{L.H.S.} = \frac{1+\omega^2+1+\omega}{(1+\omega)(1+\omega^2)}$ ✓

$= \frac{1+\omega+\omega^2+1}{1+\omega^2+\omega+\omega^3}$ ✓

$= \frac{1+\omega+\omega^2+1}{1+\omega^2+\omega+1}$ ✓

$= 1$

$= \text{R.H.S.}$

ii/ $\frac{1+\omega^n+\omega^{2n}}{3}$

Let $n = 3k$ (ie × of 3)

$\frac{1+\omega^{3k}+\omega^{6k}}{3} = \frac{1+1+1}{3} =$ ✓

ii/ $n = 3k+1$

$\frac{1+\omega^{3k+1}+\omega^{6k+2}}{3} = \frac{1+\omega+\omega^2}{3}$

$= 0$ ✓

$n = 3k+2$

$\frac{1+\omega^{3k+2}+\omega^{6k+4}}{3} = \frac{1+\omega^2+\omega}{3}$

$= 0$ ✓

$\therefore$ Statement true

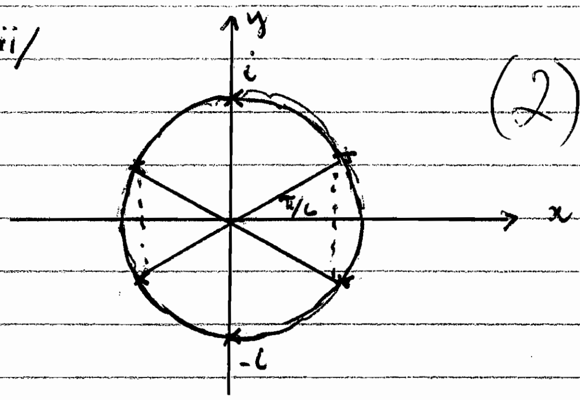
QR

Q4) a) i/ $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$z^6 = \cos\pi + i\sin\pi$ (1)

$= -1$

ii/



(2)

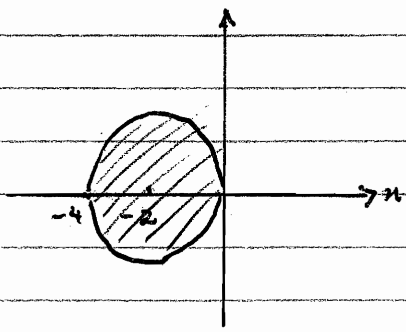
b) $z\bar{z} = |z|^2 = x^2 + y^2$

$z + \bar{z} = 2\,Re(z) = 2x$ Method (1)

$z\bar{z} + 2(z+\bar{z}) \leq 0$

$x^2 + y^2 + 4x \leq 0$

$(x+2)^2 + y^2 \leq 4$ (1)



(1)

c) i/ $OA = \omega$

(1) $\therefore OC = \frac{1}{2}i\omega$ | $2i\omega$

ii/ $OB = OA + OC$

(1) $= \omega + \frac{1}{2}i\omega$ | $\omega + 2i$

$OD = \frac{OB}{2}$

(1) $= \frac{\omega}{2} + \frac{i\omega}{4}$ | $\frac{\omega + i\omega}{2}$

iii) $z^6 - 1 = (z^2-1)(z^4+z^2+1)$

roots of $z^4 + z^2 + 1 = 0$

$z^2 = \frac{-1 \pm \sqrt{1-4}}{2}$

(1) $= \frac{-1 \pm \sqrt{3}i}{2}$

$\therefore$ roots of z

$\frac{-1 \pm \sqrt{3}i}{2}, \quad \frac{+1 \pm \sqrt{3}i}{2}$

(2) $\text{cis}\left(\pm\frac{\pi}{3}\right) \quad \text{cis}\left(\pm\frac{2\pi}{3}\right)$