NORTH SYDNEY BOYS HIGH SCHOOL

2011 YEAR 12 HSC
ASSESSMENT TASK 1

Mathematics

Extension 2

Examiner: S. Ireland

General Instructions

- Working time – **55 minutes**
- Write on both sides of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Fletcher
- O Mr Ireland
- O Mr Rezcallah

Student Number:

Question	**1**	**2**	**3**	**4**	**5**	**6**	Total	Total
Mark	/11	/9	/10	/8	/7	/8	/53	/100

Question 1 (11 marks) Start a new page. **Marks**

(a) Let z be the complex number $2\ cis(-\frac{\pi}{3})$. On a single, half-page Argand diagram, plot the points: $z, \bar{z}, z^2$, and $\frac{1}{z}$. 4

(b) Given $z = 1 - i$ and $\omega = -1 + i\sqrt{3}$,

(i) Express $\frac{1}{z}$ in $a + ib$ form (where a and b are real) 1

(ii) Calculate ωz in $a + ib$ form 1

(iii) Find arg z and arg ω 2

(iv) Hence find arg (ωz) 1

(v) Hence prove $\sin\frac{5\pi}{12} = \frac{\sqrt{2}(1+\sqrt{3})}{4}$ 2

Question 2 (9 marks) Start a new page.

(a) If ω is a non-real root of $z^3 = 1$, then

(i) Show that $1 + \omega + \omega^2 = 0$ 1

(ii) Evaluate $(1 - 3\omega + \omega^2)(1 + \omega - 8\omega^2)$ 2

(b) If $1 + i$ is a zero of $P(x) = 3x^3 - 7x^2 + 8x - 2$, then

(i) Find all zeroes of $P(x)$ 2

(ii) Factorise $P(x)$ over $\mathbb{R}$ 1

(c) Factorise $P(x) = 3x^4 + 8x^3 + 6x^2 - 1$ completely given that $P(x) = 0$ has a root of multiplicity 3. 3

Question 3 (10 marks) Start a new page.

(a) (i) Find real numbers x and y such that $(x + iy)^2 = -3 - 4i$ 2

(ii) Hence solve the equation $z^2 - 3z + (3 + i) = 0$ 2

(b) On separate Argand diagrams sketch the locus of a point which satisfies:

(i) $|z - 1| = |z - i|$ 2

(ii) $z^2 - (\bar{z})^2 = 16i$ 2

(c) Sketch the region satisfying both $0 \leq \arg(z - 2) \leq \frac{\pi}{2}$ and $\mathrm{Im}(z) \leq 1$ 2

Question 4 (8 marks) Start a new page.

Given that $x^3 - 2x^2 - 5 = 0$ has roots α, β, γ,

(a) Evaluate $\alpha^2 + \beta^2 + \gamma^2$ 2

(b) Evaluate $\alpha^3 + \beta^3 + \gamma^3$ 2

(c)Write equations with roots:

(i) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ 2

(ii) $\alpha + \beta + 2\gamma,\ \ \alpha + 2\beta + \gamma,\ \ 2\alpha + \beta + \gamma$ 2

Question 5 (7 marks) Start a new page.

(a) Sketch z_1 and z_2 on an Argand diagram given that $|z_1| = |z_2| = |z_1 - z_2|$

Hence find the magnitude of $\arg\left(\frac{z_1}{z_2}\right)$ 2

(b) (i) Sketch $|z - 15i| \leq 5$ on an Argand diagram. 2

(ii) Find the complex number z with the least positive argument satisfying $|z - 15i| \leq 5$. (Give your answer in $a + ib$ form). 3

Question 6 (8 marks) Start a new page.

(a) (i) Show the roots of $z^5 = 1$ on a unit circle on an Argand diagram. 3

(ii) Hence show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$ 2

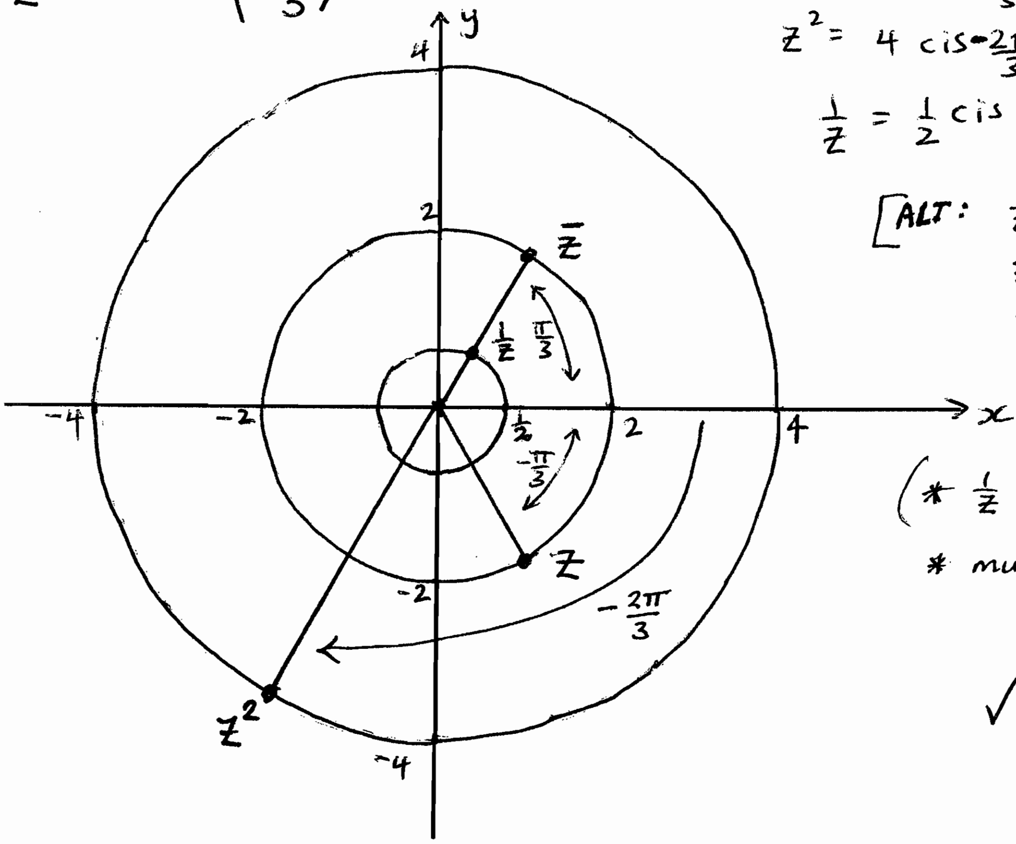
(b) Sketch the locus $\arg(1 - \frac{1}{z}) = \frac{\pi}{4}$ and find its Cartesian equation. 3

END OF EXAMINATION

2011 HSC Ext. 2 – Task 1 — SUGGESTED SOLUTIONS

Q1

(a) $z = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$



$\bar{z} = 2\,\text{cis}\,\frac{\pi}{3}$

$z^2 = 4\,\text{cis}\,-\frac{2\pi}{3}$

$\frac{1}{z} = \frac{1}{2}\,\text{cis}\,\frac{\pi}{3}$

[ALT: $z = 1 - \sqrt{3}\,i$

$z^2 = -2 - 2\sqrt{3}\,i$

$\bar{z} = -2 - 2\sqrt{3}\,i$

$\frac{1}{z} = \frac{1}{4} + \frac{\sqrt{3}}{4}\,i$]

(* $\frac{1}{z}$ & $\bar{z}$ must have same arg.

* must show data on sketch.)

✓✓✓✓

(b) (i) $\frac{1}{z} = \frac{1}{2} + \frac{1}{2}i$ ✓

(ii) $\omega z = (\sqrt{3} - 1) + i(\sqrt{3} + 1)$ ✓

(iii) $z = 1 - i \quad \therefore \arg z = -\frac{\pi}{4}$ ✓

$\omega = -1 + i\sqrt{3} \quad \therefore \arg \omega = \frac{2\pi}{3}$ ✓

(iv) $\arg(\omega z) = \frac{5\pi}{12}$ ✓

(v) $\sin\frac{5\pi}{12} = \frac{\text{Im}(\omega z)}{|\omega z|}$

$= \frac{\text{Im}(\omega z)}{|\omega| \cdot |z|}$

$= \frac{\sqrt{3} + 1}{2 \cdot \sqrt{2}}$

$\therefore \sin\frac{5\pi}{12} = \frac{\sqrt{2}(\sqrt{3} + 1)}{4}$ as required. ✓✓

[* must use previous part of question]

[ALT: $\omega z = 2\sqrt{2}\,\text{cis}\,\frac{5\pi}{12}$ & $\omega z = (\sqrt{3} - 1) + i(\sqrt{3} + 1)$, then equate imag. parts]

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Question 2:

(a) (i) $w^3 = 1$

$w^3 - 1 = 0$

$(w-1)(w^2+w+1) = 0.$ ✓

$\therefore \quad w^2 + w + 1 = 0$

(ii) $(1-3w+w^2)(1+w-8w^2)$

$= (\underbrace{1+w^2}) - 3w)(1+w-8w^2)$

$= (-w-3w)(-w^2-8w^2) = (-4w)(-9w^2)$ ✓

$= +(4w)(9w^2) = 36w^3 = 36.$ ✓ for 36

(b) (i) $1+i$ is a zero $\Rightarrow$ $1-i$ is a zero. ✓

$S = -\frac{b}{a} = \frac{7}{3}.$

$1+i+1-i+\alpha = \frac{7}{3}.$

$2 + \alpha = \frac{7}{3}.$

$\alpha = \frac{7}{3} - 2 = \frac{1}{3}.$ ✓

(ii) $P(x) = (3x-1)(x^2 - \text{Sum}\, x + P)$

Product $= P = (1+i)(1-i) = 1 - i^2 = 2.$

Sum $= S = 1+i+1-i = 2.$

$\Big\}\ x^2-2x+2.$

$P(x) = (3x-1)(x^2-2x+2).$ ✓ for $(3x-1)(x^2-2x+2)$

or:

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -7 & 8 & -2 \\ & & 1 & -2 & 2 \\ \hline & 3 & -6 & 6 & 0 \end{array}$$

N.B: $(x-\frac{1}{3})(x^2-2x+2)$ won't get the mark.

$P(x) = (x-\frac{1}{3})(3x^2-6x+6)$

$= (3x-1)(x^2-2x+2).$

(c) $P(x) = 3x^4 + 8x^3 + 6x^2 - 1.$

$P'(x) = 12x^3 + 24x^2 + 12x.$

$P''(x) = 36x^2 + 48x + 12 = 12(3x^2+4x+1)$ ✓

$= 12(3x+1)(x+1)$

Question 2 - (cont'd)

$P'(-1) = P''(-1) = 0.$

$\therefore\ x = -1$ is a triple root. ✓

$S = -\frac{b}{a} = -\frac{8}{3}.$

$-1 + -1 + -1 + \beta = -\frac{8}{3}.$

$-3 + \beta = -\frac{8}{3}.$

$\beta = 3 - \frac{8}{3} = \frac{1}{3}.$ ✓ for $(x+1)^3$

$\therefore\ P(x) = (x+1)^3(3x-1).$ ✓ for correct factors

2ND Method:

-1	3	8	6	0	-1
		-3	-5	1	1
	3	5	1	1	0

$(3x^3 + 5x^2 + x + 1)(x+1)$

-1	3	5	1	1
		-3	-2	-1
	3	2	-1	0

$(x+1)^2(3x^2 + 2x - 1)$

$P(x) = (x+1)^2(x+1)(3x-1)$

$P(x) = (x+1)^3(3x-1).$

Again $P(x) = (x+1)^3\left(x - \frac{1}{3}\right)$ will not get the mark.
If students do the same mistake twice, they are penalised once for this error.

Q3

(a) (i) $(x+iy)^2 = x^2 - y^2 + 2xyi$

$\therefore \begin{cases} x^2 - y^2 = -3 \\ xy = -2 \end{cases}$

$x^2 - \frac{4}{x^2} = -3$

$(x^2-1)(x^2+4) = 0$

$\therefore x = 1$ or -1 (x is real)

$\therefore y = -2$ or 2

So answer is $x=1, y=-2$ or $x=-1, y=2$.

(ii) $z^2 - 3z + (3+i) = 0$

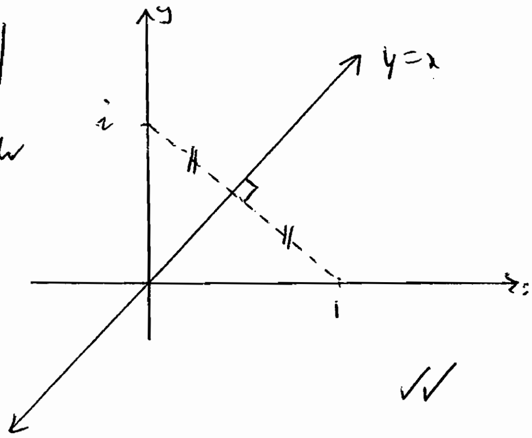
$z = \dfrac{3 \pm \sqrt{9 - 4(3+i)}}{2}$

$= \dfrac{3 \pm (1-2i)}{2}$

$\therefore z = 2 - i$ or $1 + i$

(b) (i) $|z-1| = |z-i|$

(* must show perpendicular bisector or give equation)

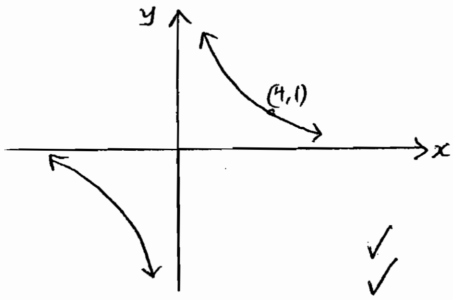


(ii) $z^2 - (\bar{z})^2 = 16i$

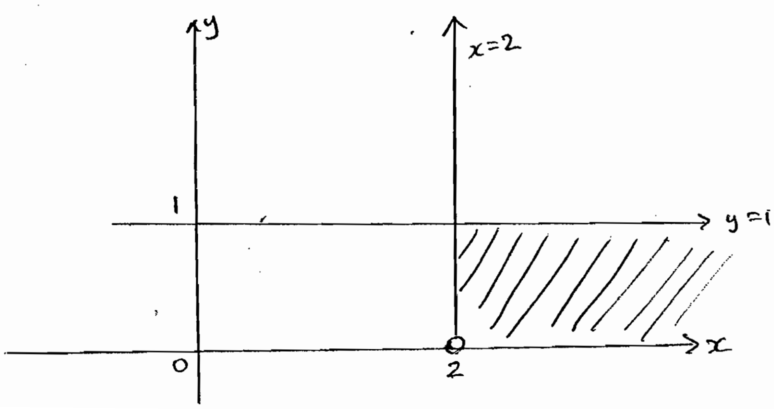
$\therefore (z - \bar{z})(z + \bar{z}) = 16i$

$2yi \cdot 2x = 16i$

$xy = 4$



(c) $0 \leq \arg(z-2) \leq \frac{\pi}{2}$ and $\text{Im}(z) \leq 1$



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Q4 $x^3 - 2x^2 - 5 = 0$

(a) $\alpha + \beta + \gamma = 2$

$\alpha\beta + \alpha\gamma + \beta\gamma = 0$

$\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ ✓

$= 2^2 - 2(0)$

$= 4$ ✓

(b) Sub. α, β, γ into equation :-

$$\begin{aligned} &\alpha^3 - 2\alpha^2 - 5 = 0 \\ + \quad &\beta^3 - 2\beta^2 - 5 = 0 \\ &\gamma^3 - 2\gamma^2 - 5 = 0 \\ \hline &\alpha^3 + \beta^3 + \gamma^3 - 2(\alpha^2 + \beta^2 + \gamma^2) - 15 = 0 \end{aligned}$$ ✓

$\therefore \alpha^3 + \beta^3 + \gamma^3 = 15 + 2(4)$

$= 23$ ✓

(c) (i) $\left(\frac{1}{x}\right)^3 - 2\left(\frac{1}{x}\right)^2 - 5 = 0$ ✓

$\frac{1}{x^3} - \frac{2}{x^2} - 5 = 0$

$\therefore 1 - 2x - 5x^3 = 0$ ✓

(ie $5x^3 + 2x - 1 = 0$)

(ii) Since $\alpha + \beta + \gamma = 2$, we want roots $\alpha + 2, \beta + 2, \gamma + 2$. ✓

$\therefore$ equation is $(x-2)^3 - 2(x-2)^2 - 5 = 0$ ✓

$\therefore x^3 - 6x^2 + 12x - 8 - 2x^2 + 8x - 8 - 5 = 0$

$\therefore x^3 - 8x^2 + 20x - 21 = 0$

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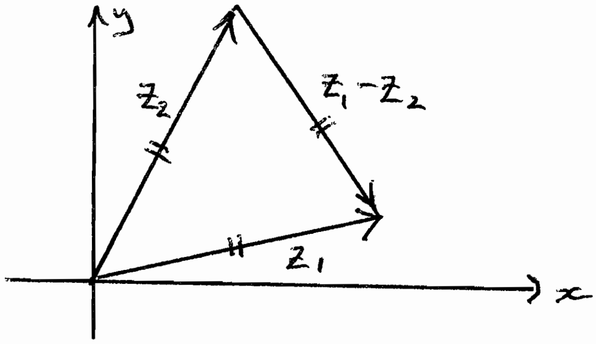
Q5.

(a)



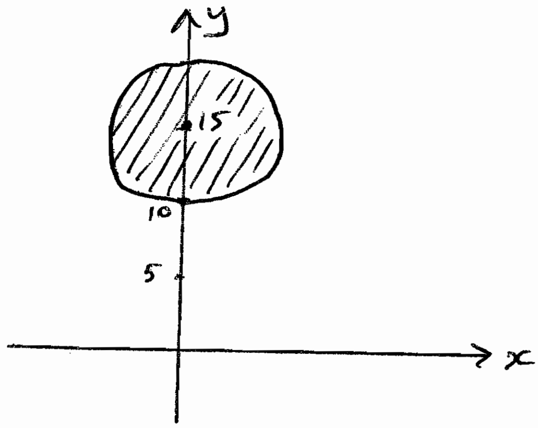
$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

$$= \pm \frac{\pi}{3} \quad \text{(as } \Delta \text{ is equilateral)}$$

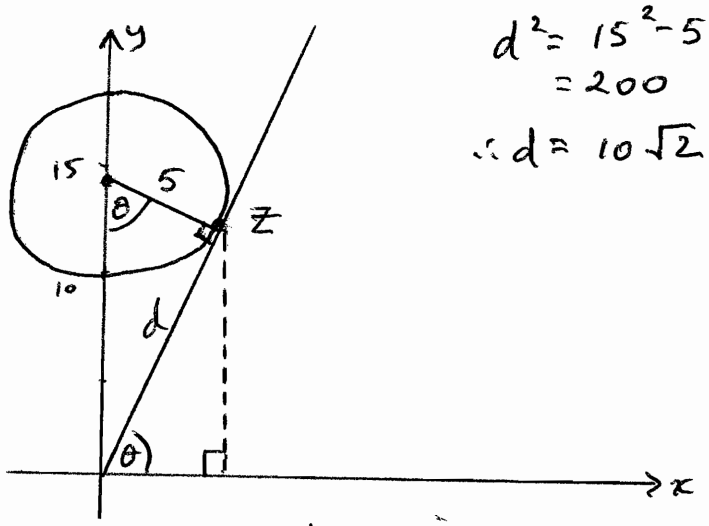
(ie magnitude $= \frac{\pi}{3}$)

(b)

(i)



(ii)



$d^2 = 15^2 - 5^2$
$= 200$
$\therefore d = 10\sqrt{2}$

$\therefore$ z has x coordinate
$= d\cos\theta$
$= 10\sqrt{2} \cdot \frac{5}{15}$
$= \frac{10\sqrt{2}}{3}$.

Likewise, z's y coordinate
$= 10\sqrt{2} \cdot \frac{10\sqrt{2}}{15}$
$= \frac{40}{3}$

$$\therefore z = \frac{10\sqrt{2}}{3} + \frac{40}{3} i$$

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Q6 - continued

(5-b) 2^{ND} Approach:

$Re\left(1-\frac{1}{z}\right) = Im\left(1-\frac{1}{z}\right)$ since angle is $\pi/4$.

Let $z = x+iy$.

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}.$$

$$1-\frac{1}{z} = 1 - \frac{(x-iy)}{(x^2+y^2)} = \frac{x^2+y^2-x+iy}{x^2+y^2}.$$

$$Re\left(1-\frac{1}{z}\right) = \frac{x^2+y^2-x}{x^2+y^2}.$$

$$Im\left(1-\frac{1}{z}\right) = +\frac{y}{x+y^2}.$$

$$\therefore \quad x^2+y^2-x = +y.$$

$$x^2-x+y^2-y = 0.$$

$$\underbrace{x^2-x+\frac{1}{4}} + y^2-y+\frac{1}{4} = \frac{1}{4}+\frac{1}{4}.$$

$$\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{2}, \quad y>0.$$

$\therefore$ Circle of centre $\left(\frac{1}{2},\frac{1}{2}\right)$ and $r=\frac{1}{\sqrt{2}}$.

Locus is a major arc of this circle.

3rd approach: $\tan^{-1}\left(1-\frac{1}{z}\right) = \tan^{-1}\pi/4.$

$$\frac{y}{x^2+y^2-x} = 1.$$

$$x^2+y^2-x = y.$$

$$x^2+y^2-x-y = 0.$$

$$\vdots$$

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2 = \frac{1}{2}, \text{ where } y>0.$$

Locus is the major arc of this circle.
(sketched before).

Question 6

(a) $z^5 = 1.$

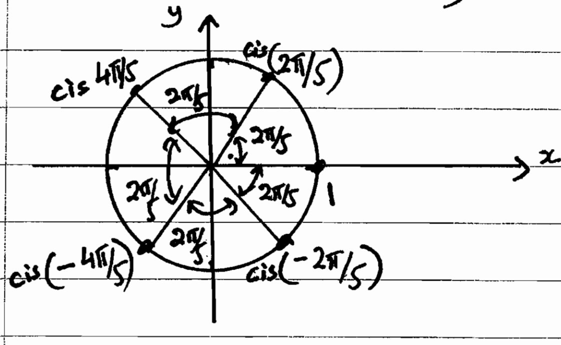
$(\cos\theta + i\sin\theta)^5 = 1$

$\cos 5\theta + i\sin 5\theta = 1.$

$5\theta = 2k\pi$

$\therefore \theta = \frac{2k\pi}{5} \qquad k = 0, \pm 1, \pm 2.$

Roots are $\text{cis}\,0 = 1$, $\text{cis}\,\frac{2\pi}{5}$, $\text{cis}\left(-\frac{2\pi}{5}\right)$, $\text{cis}\,\frac{4\pi}{5}$, $\text{cis}\left(-\frac{4\pi}{5}\right)$.



✓✓ for correct circle with the roots shown.

✓ for showing (even in ii that $\text{cis}\,8\pi/5 = \text{cis}(-2\pi/5)$, $\text{cis}\,6\pi/5 = \text{cis}(-4\pi/5)$

(ii) Adding the sum of the roots for $z^5 - 1 = 0$.

$1 + \text{cis}\,\frac{2\pi}{5} + \text{cis}(-2\pi/5) + \text{cis}\,4\pi/5 + \text{cis}(-4\pi/5) = 0.$ ✓

From diagram: $1 + \cos 2\pi/5 + \cos(-2\pi/5) + \cos 4\pi/5 + \cos(-4\pi/5) = 0.$

But $\cos(-2\pi/5) = \cos 2\pi/5 \qquad \cos(-4\pi/5) = \cos 4\pi/5.$

$1 + 2\cos(2\pi/5) + 2\cos(4\pi/5) = 0.$

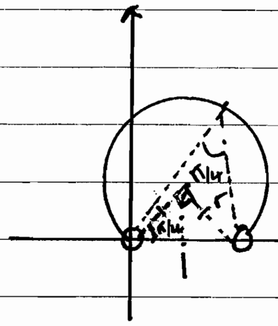
$2\cos 2\pi/5 + 2\cos 4\pi/5 = -1 \qquad \div 2.$ ✓

$\cos 2\pi/5 + \cos 4\pi/5 = -\frac{1}{2}.$

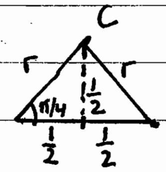
(b) $\arg\left(1 - \frac{1}{z}\right) = \pi/4.$

$\arg\left(\frac{z-1}{z}\right) = \pi/4.$

$\cos\pi/4 = \frac{r}{1} \Rightarrow r = \frac{1}{\sqrt{2}}.$



✓ for correct shape



$C\left(\frac{1}{2}, \frac{1}{2}\right).$

✓ for centre, radius or equation of circle

Equation of locus: $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$ where $y > 0$. } ✓ for correct answers!

N.B: The last mark should show $y > 0$ or equivalent in the equation