100 Years

North Sydney Boys High School

1912 ~ 2012

MATHEMATICS (EXTENSION 2)

2012 HSC Course Assessment Task 1

November 28, 2011

General instructions

- Working time – 55 minutes.
- Commence each new question on a new page. Write on both sides of the paper.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

Class (please ✔)

- ◯ 12M4A – Mr Weiss
- ◯ 12M4B – Mr Ireland
- ◯ 12M4C – Mr Fletcher

NAME: .... **# BOOKLETS USED**: ....

Marker's use only.

QUESTION	1	2	3	Total	%
MARKS	20	20	18	58	

Question 1 (20 Marks) Commence a NEW page. **Marks**

(a) i. Find the values of the real numbers a and b such that $(a+ib)^2 = 5-12i$. **2**

ii. Hence or otherwise, solve the equation $ix^2+3x+(3-i)=0$. **2**

(b) Evaluate $(1+i)^9$. **2**

(c) On an Argand diagram, sketch the locus of a point which satisfies

i. $|z-i|=4$. **2**

ii. $(z-1)(\overline{z}+1)=3$. **2**

iii. $|z+2|=|z-2|$. **2**

iv. $\operatorname{Arg}\left(\dfrac{z+3i}{z-1}\right)=\dfrac{\pi}{6}$. **2**

(d) i. If $z_1=3+4i$ and $|z_2|=13$, find the greatest value of $|z_1+z_2|$. **3**

ii. If $|z_1+z_2|$ has its greatest value and also $0<\arg z_2<\frac{\pi}{2}$, express z_2 in the form $a+ib$ where a and b are real. **3**

Question 2 (20 Marks) Commence a NEW page. **Marks**

(a) If ω represents one of the complex cube roots of unity, evaluate **3**

$$\left(1-\omega^8\right)\left(1-\omega^4\right)\left(1-\omega^2\right)(1-\omega)$$

(b) In an Argand diagram, the points P, Q and R represent the complex numbers z_1, z_2 and $z_2 + i\,(z_2 - z_1)$ respectively.

i. Show that $\triangle PQR$ is right-angled. **2**

ii. Find in terms of z_1 and z_2 the complex number represented by the point S such that $PQRS$ is a rectangle. **2**

(c) i. Use De Moivre's Theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **2**

ii. Deduce that $8x^3 - 6x - 1 = 0$ has solutions $x = \cos\theta$, where $\cos 3\theta = \frac{1}{2}$. **3**

iii. Find the roots of $8x^3 - 6x - 1 = 0$ in the form $\cos\theta$. **2**

iv. Hence evaluate $\cos\dfrac{\pi}{9}\cos\dfrac{2\pi}{9}\cos\dfrac{4\pi}{9}$. **2**

(d) For the complex numbers

$$z_1 = (1+i)^2 \qquad \text{and} \qquad z_2 = \sqrt{2}\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right]$$

i. Express z_1 in modulus-argument form. **1**

ii. Express z_2, $\overline{z_2}$ and iz_1 in Cartesian form. **3**

Question 3 (18 Marks) Commence a NEW page. **Marks**

(a) i. Use the remainder theorem to find a factor of $x(x-1) - a(a-1)$. **1**

ii. By division or otherwise, find the other factor. **2**

(b) Prove that if the polynomial $P(x)$ has a root of multiplicity m then $P'(x)$ has a root of multiplicity $(m-1)$. **3**

(c) Given $(x-2)^2$ is a factor of $x^3 - 3x^2 + ax + b$, find the value of a and b and hence factorise the polynomial over $\mathbb{C}$. **4**

(d) The equation $x^3 + 3x^2 + 4x - 7 = 0$ has roots α, β and γ. Find the equation with roots 2α, 2β and 2γ. **3**

(e) If $(1+i)$ is a root of the equation $x^3 - ax^2 + bx - 4 = 0$ where a, $b \in \mathbb{R}$, find the values of a and b and hence solve the equation. **5**

End of paper.

2012 HSC - Ext. 2 - Task 1

SUGGESTED SOLUTIONS

Q1

(a) (i) $(a+ib)^2 = 5-12i$

$\therefore \begin{cases} a^2-b^2=5 \\ 2ab=-12 \end{cases}$

$\therefore a^2 - \left(\frac{6}{a}\right)^2 = 5$

$a^4 - 5a^2 - 36 = 0$ ✓

$(a^2-9)(a^2+4) = 0$

$\therefore a = \pm 3$ (a is real)

$\therefore a=3,\ b=-2$ or $a=-3,\ b=2$. ✓

[ALT. $a^2-b^2=5$
$ab=-6$
$\therefore$ by inspection, $a=3, b=-2$ ✓ or $a=-3, b=2$ ✓].

(Note: answer can also be written as $\pm(3-2i)$.
But "$a=\pm 3,\ b=\pm 2$" loses a mark.)

(ii) $ix^2 + 3x + (3-i) = 0$

$\therefore x = \dfrac{-3 \pm \sqrt{9-4i(3-i)}}{2i}$

$= \dfrac{-3 \pm \sqrt{5-12i}}{2i}$ ✓

$= \dfrac{-3 \pm (3-2i)}{2i}$

$= \dfrac{-2i}{2i}$ or $\dfrac{-6+2i}{2i}$

$= -1$ or $3i+1$ ✓

(* answers must be simplified for full marks:
ie no i's on denominator)

Q1 - continued

(b) $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right) = \sqrt{2}\ \mathrm{cis}\frac{\pi}{4}$

$\therefore (1+i)^9 = (\sqrt{2})^9\ \mathrm{cis}\frac{9\pi}{4}$ ✓

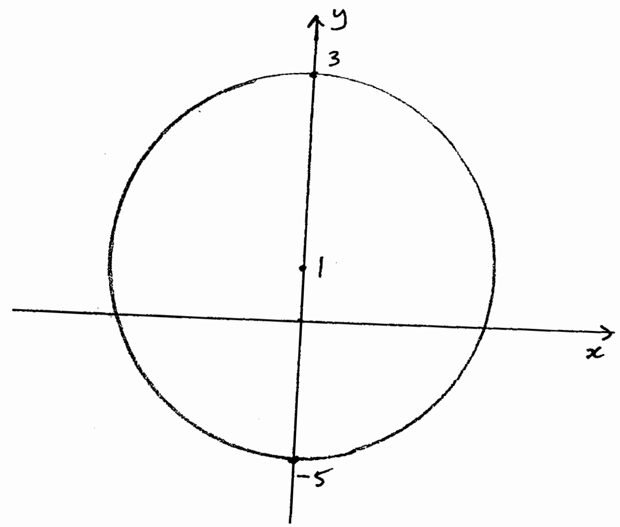
$= 16\sqrt{2}\ \mathrm{cis}\frac{\pi}{4}$

$= 16(1+i) = 16+16i$ ✓

[ALT. $(1+i)^2 = 2i$
$\therefore (1+i)^8 = (2i)^4 = 16$
$\therefore (1+i)^9 = 16(1+i) = 16+16i$]

(c) (i) $|z-i| = 4 \quad \therefore |z-(0+i)| = 4$

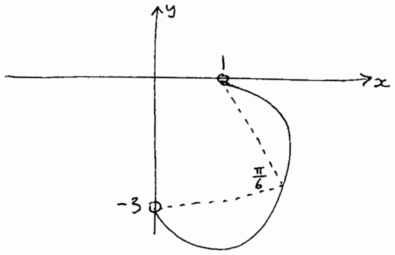
This is a circle, centre $(0,1)$, radius 4



✓✓

Q1 – continued

(iv) $\operatorname{Arg}\left(\frac{z+3i}{z-1}\right) = \frac{\pi}{6}$



(* Locus is a major arc; in Q4; and endpoints are open circles at (0,-3) & (1,0).)

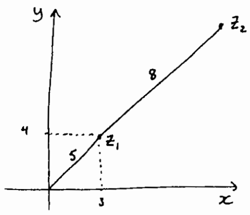
(d) (i) $|z_1 + z_2| \leq |z_1| + |z_2|$ (Δ inequality)

$= \sqrt{3^2+4^2} + 13$

$= 18$

i.e. max. value is 18

(ii) $|z_1 + z_2|$ has its max. when z_1 & z_2 are collinear.



Thus $z_2 = k z_1$

$= k(3+4i)$

$\therefore (3k)^2 + (4k)^2 = 13^2$

$\therefore k = \frac{13}{5}$

$\therefore z_2 = \frac{13}{5}(3+4i)$

$= \frac{39}{5} + \frac{52}{5}i$.

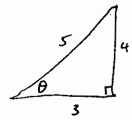
(alt. →)

Q1 – continued.

[(d) (ii) ALT.: z_1 & z_2 collinear.

$\therefore \arg z_2 = \arg z_1 = \tan^{-1}\left(\frac{4}{3}\right) = \theta$

[* as $0 < \arg z_2 < \frac{\pi}{2}$]



$\therefore \sin\theta = \frac{4}{5}$

& $\cos\theta = \frac{3}{5}$

$\therefore z_2 = 13(\cos\theta + i\sin\theta)$

$= \frac{39}{5} + \frac{52}{5}i$, as before.]

20

Q1 – continued

(c) (ii) $(z-1)(\bar{z}+1) = 3$

$\therefore z\bar{z} - \bar{z} + z - 1 = 3$

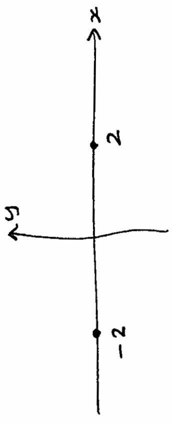
$\therefore x^2+y^2 - (x-iy) + (x+iy) - 1 = 3$

$\therefore x^2+y^2 + 2yi = 4$

Equating Re and Im parts,

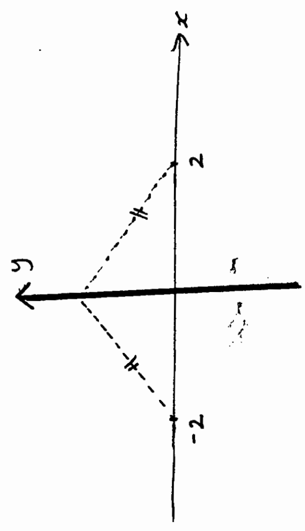
$\therefore x^2+y^2 = 4$ and $y = 0$

Thus the locus comprises 2 points :



(iii) $|z+2| = |z-2|$

The locus is all those points equally distant from (-2,0) and (2,0) i.e. the y-axis $(x=0)$.



(* axes are difficult to show when a locus, So best to write the equation $x=0$.)

Q3) (a)

$1+\omega+\omega^2=0$

$1=-\omega-\omega^2$

$\omega^3=1$

$(1-\omega^8)(1-\omega^4)(1-\omega^2)(1-\omega) = (1-(\omega^3)^2\omega^2)(1-\omega^4)(1-\omega^2)(1-\omega)$

$= (1-\omega^2)(1-\omega)(1-\omega^2)(1-\omega)$

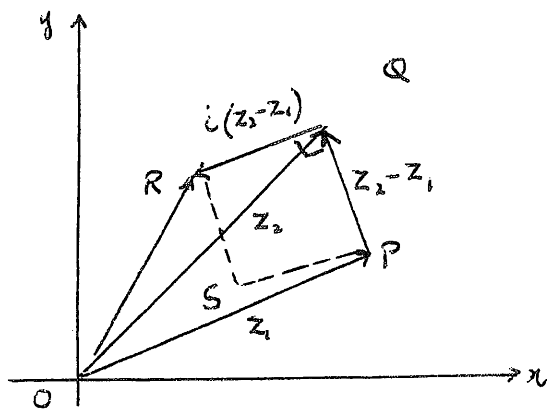
$= \{(1-\omega^2)(1-\omega)\}^2$

$= (1-\omega-\omega^2+\omega^3)^2$

$= (1+1+1)^2$

$= 9$

(b)



i) $P\hat{Q}R = \frac{\pi}{2}$

$\therefore \triangle PQR$ is right angled at Q

ii) If PQRS is a rectangle

$\vec{PS} \parallel \vec{QR}$

and hence $\vec{PS}$ also represents $i(z_2-z_1)$

Now $\vec{OS}$ is the vector sum of $\vec{OP}$ and $\vec{PS}$

Hence S represents $z_1 + i(z_2-z_1)$

(c) i/ $(\cos\theta+i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

also $(\cos\theta+i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$\therefore \cos 3\theta = \cos^3\theta - 3\cos(1-\cos^2\theta)$

$= 4\cos^3\theta - 3\cos\theta$

ii/ Now $8\cos^3\theta - 6\cos\theta - 1 = 0$

$\therefore 4\cos^3\theta - 3\cos\theta = \frac{1}{2}$

$\therefore \cos 3\theta = \frac{1}{2}$

$\therefore 3\theta = \frac{\pi}{3} \pm 2n\pi$

iii/ $\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \ldots$

$\therefore$ roots of $8x^3-6x+1=0$

are $\cos\frac{\pi}{9}$ $\cos\frac{5\pi}{9}$, $\cos\frac{7\pi}{9}$

$\cos\frac{5\pi}{9} = -\cos\frac{4\pi}{9}$

iv/ $\cos\frac{7\pi}{9} = -\cos\frac{2\pi}{9}$

$\therefore$

$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$

(d) i/ $z_1 = (1+i)^2$

$= 2i$

$= 2\operatorname{cis}\frac{\pi}{2}$

ii/ $z_2 = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

$= -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}i}{2}$

$\overline{z_2} = -\frac{\sqrt{6}}{2} + \frac{\sqrt{2}i}{2}$

$iz_1 = i \times 2i$

$= -2$

Q3) (a) i/ $f(x) = x(x-1) - a(a-1)$

$f(a) = a(a-1) - a(a-1)$

$= 0$

$\therefore (x-a)$ is a factor

ii/

$$\begin{array}{r} x \quad + (a-1) \\ x-a \;\big)\overline{\; x^2 - x - a^2 + a} \\ \underline{x^2 - ax} \\ ax - x \\ ax - x - a^2 + a \end{array}$$

$\therefore f(x) = (x-a)(x+a-1)$

(c) $P(x) = x^3 - 3x^2 + ax + b$

$P'(x) = 3x^2 - 6x + a$

$P'(2) = 0$ because $(x-2)$ is a double factor

$0 = 12 - 12 + a \Rightarrow a = 0$

$P(2) = 0$

$0 = 8 - 12 + 0 + b \Rightarrow b = 4$

$\therefore a = 0 \quad b = 4$

$P(x) = (x-2)^2(x+1)$

(d) If $x = 2\alpha, 2\beta, 2\delta$

then $\frac{x}{2} = \alpha, \beta, \delta$

$x^3 + 3x^2 + 4x - 7 = 0$

thus

$\left(\frac{x}{2}\right)^3 + 3\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right) - 7 = 0$

$\frac{x^3}{8} + \frac{3x^2}{4} + 2x - 7 = 0$

$x^3 + 6x^2 + 16x - 56 = 0$

(e) If $1+i$ is a root so is $1-i$

Let the other root be α

then $1+i + 1- + \alpha = a$

$\alpha + 2 = a$

$(1+i)(1-i)\alpha = 4$

$2\alpha = 4$

$\alpha = 2$

$\therefore a = 4$

$\therefore$ roots are $1+i, 1-i, 2$

$(1+i)(1-i) + 2(1+i) + 2(1-i) = b$

$b = 6$

(b) Let the root of multiplicity m be α

$P(x) = (x-\alpha)^m \cdot Q(x) \quad Q(\alpha) \neq 0$

$\therefore P'(x) = m(x-\alpha)^{m-1} \cdot Q(x) + (x-\alpha)^m Q'(x)$

$= (x-\alpha)^{m-1}\{m\,Q(x) + (x-\alpha)Q'(x)\}$

$= (x-\alpha)^{m-1} \cdot S(x)$

$\therefore P'(x)$ has a root of mult. $(m-)$