NORTH SYDNEY BOYS HIGH SCHOOL

100 Years 1912 ~ 2012

MATHEMATICS (EXTENSION 2)

2013 HSC Course Assessment Task 1

November 30, 2012

General instructions

- Working time – 1 hour. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided on page 3.

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12M4A – Mr Fletcher ◯ 12M4B – Mr Lam ◯ 12M4C – Ms Ziaziaris

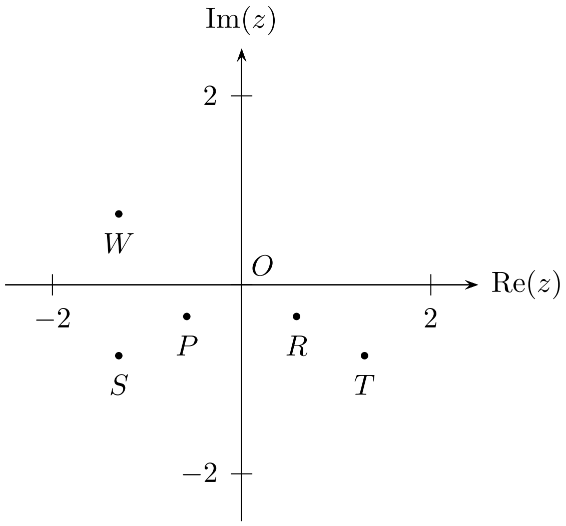
Marker's use only.

QUESTION	1 – 5	6	7	8	**Total**	%
MARKS	$\overline{5}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{50}$	

Section I: Objective response

Mark your answers on the multiple choice sheet provided. **Marks**

1. The point W on the Argand diagram below represents a number w where $|w| = 1.5$. **1**
The number w^{-1} is best represented by



(A) P

(B) R

(C) S

(D) T

2. Which of the following gives the value of z if $z^2 = 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$? **1**

(A) $\sqrt{3} + i$ or $-\sqrt{3} - i$

(B) $1 - \sqrt{3}i$ or $-1 + \sqrt{3}i$

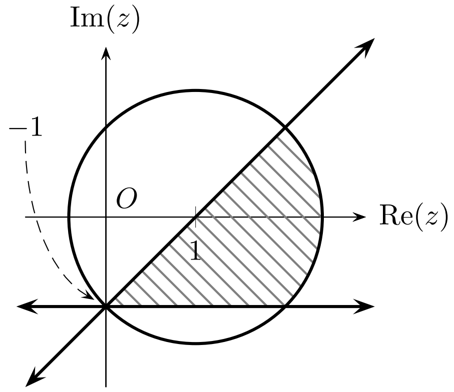
(C) $\sqrt{3} - i$ or $\sqrt{3} + i$

(D) $1 - \sqrt{3}i$ or $1 + \sqrt{3}i$

3. Let $z = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$. What is the imaginary part of $z - i$? **1**

(A) $-\frac{i}{2}$ (B) $-\frac{3i}{2}$ (C) $-\frac{1}{2}$ (D) $-\frac{3}{2}$

4. What inequality given could define the shaded area? **1**



(A) $|z-1| \leq \sqrt{2}$ and $0 \leq \operatorname{Arg}(z-i) \leq \dfrac{\pi}{4}$.

(B) $|z-1| \leq \sqrt{2}$ and $0 \leq \operatorname{Arg}(z+i) \leq \dfrac{\pi}{4}$.

(C) $|z-1| \leq 1$ and $0 \leq \operatorname{Arg}(z-i) \leq \dfrac{\pi}{4}$.

(D) $|z-1| \leq 1$ and $0 \leq \operatorname{Arg}(z+i) \leq \dfrac{\pi}{4}$.

5. Which of the following statement(s) is false, given $z = a + ib$ where $a \neq 0$ and $b \neq 0$? **1**

(A) $z - \overline{z} = 2bi$

(B) $|z|^2 = |z|\,|\overline{z}|$

(C) $|z| + |\overline{z}| = |z + \overline{z}|$

(D) $\operatorname{Arg}(z) + \operatorname{Arg}(\overline{z}) = 0$

Answer grid for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

1 – Ⓐ Ⓑ Ⓒ Ⓓ
2 – Ⓐ Ⓑ Ⓒ Ⓓ
3 – Ⓐ Ⓑ Ⓒ Ⓓ
4 – Ⓐ Ⓑ Ⓒ Ⓓ
5 – Ⓐ Ⓑ Ⓒ Ⓓ

End of Section I.
Examination continues overleaf.

Section II: Short answer

Glossary

- $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ – set of all integers.
- $\mathbb{Z}^+$ – all positive integers (excludes zero)
- $\mathbb{R}$ – set of all real numbers

Question 6 (15 Marks) Commence a NEW page. **Marks**

(a) Find the exact value of a and b if $\dfrac{4+3i}{1+\sqrt{2}i} = a + ib$ such that $a,\, b \in \mathbb{R}$. **2**

(b) i. Express $z = 1 + i\sqrt{3}$ in modulus-argument form. **2**

ii. Hence show that $z^{10} + 512z = 0$. **2**

(c) Given $z_1 = 4\left(\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}\right)$ and $z_2 = 2\left(\cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}\right)$

i. On an Argand diagram, draw the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ representing z_1, z_2 and $z_1 + z_2$ respectively. **2**

ii. Hence or otherwise, find $|z_1 + z_2|$ in simplest exact form. **2**

(d) i. Solve $z^4 + 1 = 0$, giving your answers in modulus-argument form. **3**

ii. Plot these solutions on the Argand diagram. **1**

iii. Find the exact area of the quadrilateral that they form. **1**

Question 7 (15 Marks) Commence a NEW page. **Marks**

(a) i. If $z = \cos\theta + i\sin\theta$, show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. **1**

ii. Given that $z + \dfrac{1}{z} = \sqrt{2}$, find the value of $z^{10} + \dfrac{1}{z^{10}}$. **2**

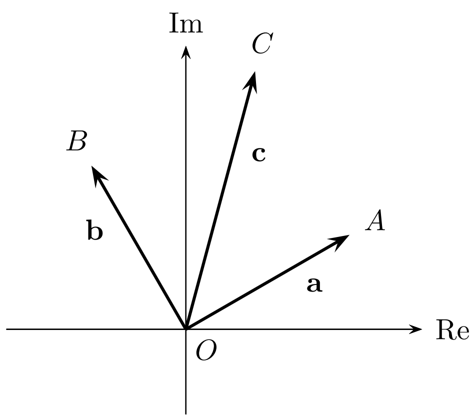
(b) Given that 1, ω and ω^2 are the cube roots of unity, find the value of $\left(1 + 2\omega + 3\omega^2\right)\left(1 + 2\omega^2 + 3\omega\right)$. **3**

(c) i. Solve $w^2 = -11 - 60i$ for w, writing your answer in the form $w = x + iy$ where $x, y \in \mathbb{R}$. **2**

ii. Hence or otherwise, solve the equation **3**

$$z^2 - (1 + 4i)z - (1 - 17i) = 0$$

(d) In the Argand diagram below, vectors **a**, **b**, **c** represent the complex numbers z_1, z_2 and $z_1 + z_2$ respectively, where $z_1 = \cos\theta + i\sin\theta$ and $z_1 + z_2 = (1 + i)z_1$.



i. Express z_2 in terms of z_1, and show that $OACB$ is a square. **2**

ii. Show that $(z_1 + z_2)\overline{(z_1 - z_2)} = 2i$. **2**

Question 8 (15 Marks) Commence a NEW page. **Marks**

(a) i. On the same diagram, sketch the locus of both $|z-2|=2$ and $|z|=|z-4i|$. **2**

ii. What is the complex number represented by the point of intersection of these two loci? **1**

(b) i. On an Argand diagram, sketch the locus of the point P representing z such that **2**

$$\left|z-\left(\sqrt{3}+i\right)\right|=1$$

ii. Find the set of possible values of $|z|$ and the set of possible values for $\operatorname{Arg} z$. **2**

(c) Let $z=x+iy$, where $x,\ y\in\mathbb{R}$ be the complex number satisfying the inequality **4**

$$z\overline{z}+(1-2i)z+(1+2i)\overline{z}\leq 4$$

Sketch the locus of z on an Argand diagram.

(d) $\arg(z-2)=\arg(z+2)+\dfrac{\pi}{4}$ is the locus of the point P representing z on an Argand diagram.

i. Show with a diagram why this locus is an arc of a circle. **2**

ii. Find the centre and radius of this circle. **2**

End of paper.

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Suggested Solutions

Section I

1. (A) **2.** (B) **3.** (C) **4.** (B) **5.** (C)

Section II

Question 6 (Ziaziaris)

(a) (2 marks)

$$\begin{aligned}
&\frac{4+3i}{1+\sqrt{2}i} \times \frac{1-\sqrt{2}i}{1-\sqrt{2}i} \\
&= \frac{4+3\sqrt{2}+i\left(3-4\sqrt{2}\right)}{1+2} \\
&= \frac{4+3\sqrt{2}}{3} + i\left(\frac{3-4\sqrt{2}}{3}\right)
\end{aligned}$$

$$a = \frac{4+3\sqrt{2}}{3} \qquad b = \left(\frac{3-4\sqrt{2}}{3}\right)$$

(b) i. (2 marks)

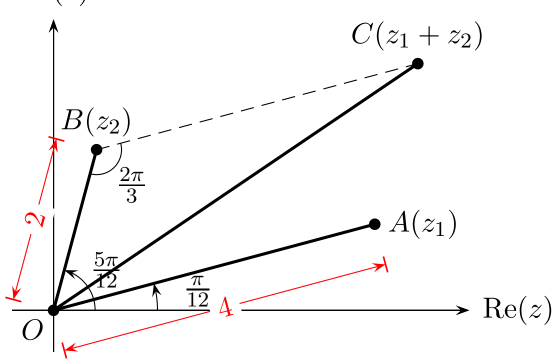
$$z = 1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

ii. (2 marks)

$$\begin{aligned}
z^{10} &= 2^{10}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{10} \\
&= 2^{10}\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right) \\
&= 2^{10}\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right) \\
512z &= 512 \times 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\
&= 2^{10}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)
\end{aligned}$$

$$\begin{aligned}
&\therefore z^{10} + 512z \\
&= 2^{10}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2} + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= 0
\end{aligned}$$

(c) i. (2 marks)



ii. (2 marks) In $OACB$,

$$\angle AOB = \frac{4\pi}{12} = \frac{\pi}{3}$$

As $OACB$ is a parallelogram, then $\angle OBC = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$. Applying the cosine rule in $\triangle OBC$,

$$\begin{aligned}
OC^2 &= |z_1 + z_2| \\
&= 2^2 + 4^2 - 2(2)(4)\cos\frac{2\pi}{3} \\
&= 20 - 8 \times \left(-\frac{1}{2}\right) = 28
\end{aligned}$$

$$\therefore OC = |z_1 + z_2| = \sqrt{28} = 2\sqrt{7}$$

(d) i. (3 marks)

$$\begin{aligned}
z^4 + 1 &= 0 \\
z^4 &= -1
\end{aligned}$$

Letting $z = \cos\theta + i\sin\theta$,

$$\begin{aligned}
&(\cos\theta + i\sin\theta)^4 \\
&= \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)
\end{aligned}$$

By De Moivre's Theorem,

$$\begin{aligned}
&\therefore \cos\theta + i\sin\theta \\
&= \cos\left(\frac{\pi + 2k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right)
\end{aligned}$$

where $k = 0, 1, 2, 3$

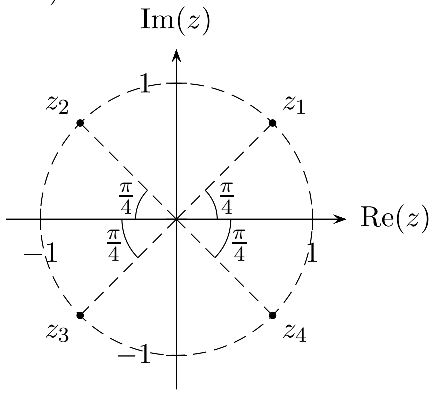
$$z_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \qquad k = 0$$

$$z_2 = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) \qquad k = 1$$

$$z_3 = \cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right) \qquad k = 2$$

$$z_4 = \cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right) \qquad k = 3$$

ii. (1 mark)



iii. (1 mark) Shape formed is a rhombus, with diagonals of length 2

$$A = \frac{1}{2}xy = \frac{1}{2} \times 2 \times 2 = 2$$

Question 7 (Fletcher)

(a) i. (1 mark)

$$\begin{aligned} z &= \cos\theta + i\sin\theta \\ \therefore z^n &= \cos n\theta + i\sin n\theta \\ z^{-n} &= \cos(-n\theta) + i\sin(-n\theta) \\ &= \cos n\theta - i\sin n\theta \\ \therefore z^n + z^{-n} &= 2\cos(n\theta) \end{aligned}$$

ii. (2 marks)

$$\begin{aligned} z + \frac{1}{z} &= \sqrt{2} = 2\cos\theta \\ \therefore \cos\theta &= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \\ \therefore \theta &= \frac{\pi}{4} \end{aligned}$$

Now,

$$\begin{aligned} z^{10} + \frac{1}{z^{10}} &= 2\cos(10\theta) \\ \therefore 2\cos\left(10 \times \frac{\pi}{4}\right) &= 2\cos\left(\frac{5\pi}{2}\right) \\ &= 2\cos\frac{\pi}{2} = 0 \end{aligned}$$

(b) (3 marks)
1, ω, ω^2 are cube roots of unity, i.e.

$$\begin{aligned} \omega^3 - 1 &= 0 \\ (\omega - 1)\left(\omega^2 + \omega + 1\right) &= 0 \\ \therefore \omega^2 &= -\omega - 1 \end{aligned}$$

Also, $\omega^3 = 1$. Use these properties to reduce higher powers of ω to lower powers:

$$\begin{aligned} &\left(1 + 2\omega + 3\omega^2\right)\left(1 + 2\omega^2 + 3\omega\right) \\ &= \left(\cancel{1 + \omega + \omega^2} + \omega + 2\omega^2\right) \\ &\quad \left(\cancel{1 + \omega + \omega^2} + \omega^2 + 2\omega\right) \\ &= \left(2\omega^2 + \omega\right)\left(\omega^2 + 2\omega\right) \\ &= \omega^2(2\omega + 1)(\omega + 2) \\ &= \omega^2\left(2\omega^2 + 5\omega + 2\right) \\ &= 2\omega^4 + 5\omega^3 + 2\omega^2 \\ &= 2\omega\cancel{\omega^3} + 5\cancel{\omega^3} + 2\omega^2 \\ &= 2\omega + 5 + 2(-\omega - 1) \\ &= 5 - 2 = 3 \end{aligned}$$

(c) i. (2 marks)

$$\omega^2 = -11 - 60i$$

Let $\omega = x + iy$

$$\begin{aligned} \therefore (x + iy)^2 &= -11 - 60i \\ x^2 - y^2 + i(2xy) &= -11 - 60i \end{aligned}$$

Equating real and imaginary parts,

$$\begin{cases} x^2 - y^2 = -11 & (1) \\ 2xy = -60 & (2) \end{cases}$$

From (2),

$$\begin{aligned} xy &= -30 \\ y &= -\frac{30}{x} \\ \therefore x^2 - \left(\frac{-30}{x}\right)^2 &= -11 \\ x^2 - \frac{900}{x^2} &= -11 \\ x^4 - 900 &= -11x^2 \\ x^4 + 11x^2 - 900 &= 0 \end{aligned}$$

Letting $u = x^2$,

$$\begin{aligned} u^2 + 11u - 900 &= 0 \\ (u + 36)(u - 25) &= 0 u = -36, 25 \\ \therefore x^2 &= 25 \\ x = \pm 5 \quad y &= \mp 6 \\ \therefore \omega &= \pm(5 - 6i) \end{aligned}$$

ii. (3 marks)

$$z^2-(1+4i)z-(1-17i)=0$$

$$\begin{aligned}z&=\frac{(1+4i)\pm\sqrt{(1+4i)^2+4(1-17i)}}{2}\\&=\frac{(1+4i)\pm\sqrt{1+8i+16i^2+4(1-17i)}}{2}\\&=\frac{(1+4i)\pm\sqrt{-15+8i+4-68i}}{2}\\&=\frac{(1+4i)\pm\sqrt{-11-60i}}{2}\\&=\frac{(1+4i)\pm(5-6i)}{2}\end{aligned}$$

Positive solution:

$$\begin{aligned}z&=\frac{1+4i+5-6i}{2}\\&=\frac{6-2i}{2}=3-i\end{aligned}$$

Negative solution:

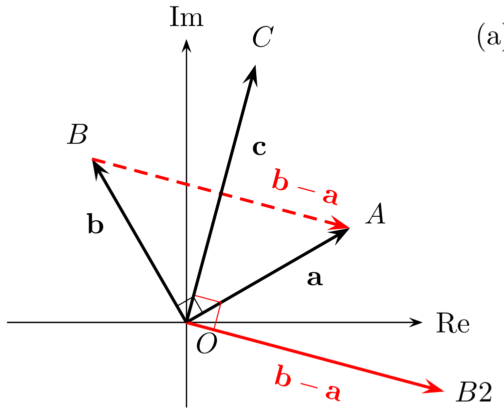
$$\begin{aligned}z&=\frac{1+4i-(5-6i)}{2}\\&=\frac{-4+10i}{2}=-2+5i\end{aligned}$$

(d) i. (2 marks)

$$\begin{aligned}z_1+z_2&=(1+i)z_1\\z_1+z_2&=z_1+iz_1\\\therefore z_2&=iz_1\end{aligned}$$

- Hence z_2 is a rotation of z_1 by 90°, i.e. $\angle AOB=\frac{\pi}{2}$.
- As $OA=OB$ and $\angle AOB=90^\circ$, $OACB$ is a square.

ii. (2 marks)



- As $\mathbf{c}$ is the diagonal of a square, then C will be representing the complex number $\sqrt{2}\,(\cos\theta+i\sin\theta)$.
- Also, $\mathbf{b}-\mathbf{a}$ is also the diagonal of the square $OACB$, it will also have modulus $\sqrt{2}$. In addition, it is a rotation of -90° of the vector $\mathbf{c}$, hence $\mathbf{b}-\mathbf{a}$ (representing the complex number z_1-z_2) is

$$\sqrt{2}\left(\cos\left(\theta-\frac{\pi}{2}\right)+i\sin\left(\theta-\frac{\pi}{2}\right)\right)$$

- $\overline{z_1-z_2}$ is therefore

$$\begin{aligned}&\sqrt{2}\left(\cos\left(\theta-\frac{\pi}{2}\right)-i\sin\left(\theta-\frac{\pi}{2}\right)\right)\\&=\sqrt{2}\left(\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right)\end{aligned}$$

- Multiplying z_1+z_2 with $\overline{z_1-z_2}$,

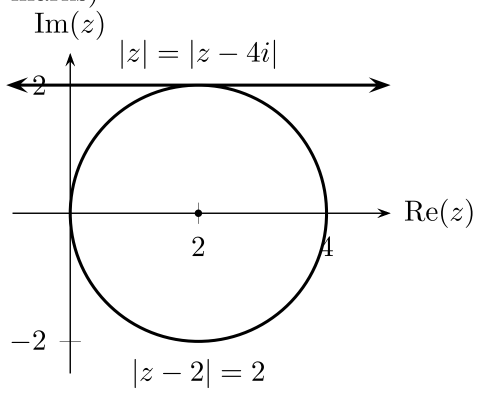
$$\begin{aligned}&\sqrt{2}\,(\cos\theta+i\sin\theta)\\&\times\sqrt{2}\left(\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right)\\&=2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\\&=2i\end{aligned}$$

Alternatively, expand via algebraic method:

$$\begin{aligned}z_2&=iz_1\\(z_1+z_2)\left(\overline{z_1-z_2}\right)&=(z_1+iz_1)\left(\overline{z_1-iz_1}\right)\\&=z_1(1+i)\overline{z_1}\,\overline{(1-i)}\\&=z_1\overline{z_1}(1+i)^2\\&=\underbrace{|z_1|^2}_{=1}\times 2i=2i\end{aligned}$$

Question 8 (Lam)

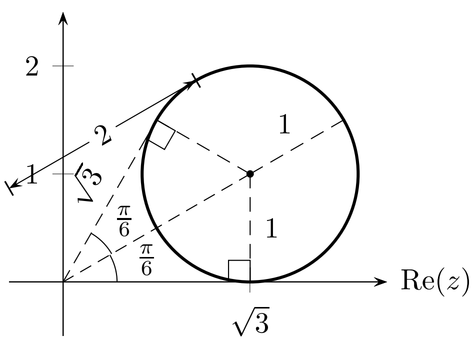
(a) i. (2 marks)



ii. (1 mark)

$$2+2i$$

(b) i. (2 marks)



ii. (2 marks)

$$0 \le \operatorname{Arg} z \le \frac{\pi}{3} \qquad 1 \le |z| \le 3$$

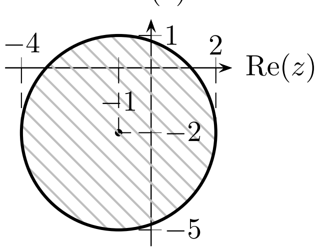
(c) (4 marks)

$$z\overline{z} + (1-2i)z + (1+2i)\overline{z} \le 4$$

Let $z = x + iy$. Then $\overline{z} = x - iy$:

$$\begin{aligned} &(x+iy)(x-iy) \\ &\quad + (1-2i)(x+iy) + (1+2i)(x-iy) \\ &= x^2 + y^2 + x + 2y + i(-2x+y) \\ &\quad + x + 2y + i(2x-y) \\ &= x^2 + y^2 + 2x + 4y \\ &= \left(x^2+2x+1\right) + \left(y^2+4y+4\right) - 5 \le 4 \\ &\quad \therefore (x+1)^2 + (y+2)^2 \le 9 \end{aligned}$$

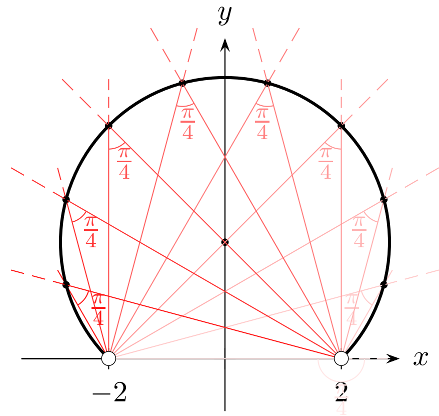
Locus is the disc with centre $(-1,-2)$ and $r = 3$:



(d) i. (2 marks)

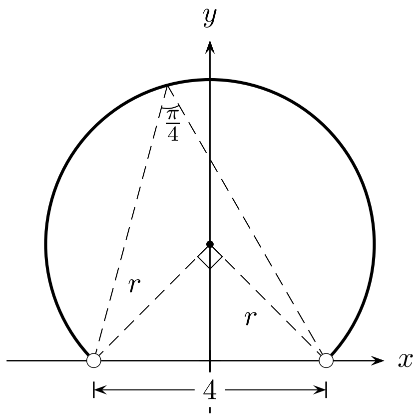
$$\arg(z-2) - \arg(z+2) = \frac{\pi}{4}$$

- $\arg(z-2) = \theta$ results in a straight line, commencing at $x = 2$ where the line makes an angle of θ with the positive x axis (similarly for $\arg(z+2) = \phi$).
- Angle between the line from $z = 2$ & $z = -2$ respectively is $\frac{\pi}{4}$.



- Hence the locus is a major arc of a circle above the x axis with AB being a chord of the circle, excluding $z = -2$ and $z = 2$.

ii. (2 marks)



- One of these "angles at the circumference" will be made by the diameter of the circle. Hence the angle at the centre of the circle will be $\frac{\pi}{2}$.
- By Pythagoras' Theorem on the right angled triangle,

$$\begin{aligned} r^2 + r^2 &= 4^2 \\ 2r^2 &= 16 \\ r^2 &= 8 \\ \therefore r &= 2\sqrt{2} \qquad C\,(0,2) \end{aligned}$$