Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2014
ASSESSMENT TASK 1

Mathematics Extension 2

General Instructions

- **Working time – 50 minutes**
  Reading time – 5 minutes
- Write on both sides of the lined paper in the booklet provided
- Write using blue or black pen
- Board approved calculators only
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.** (The multiple choice section is answered in a grid within the question booklet)
- Attempt all questions.

Class Teacher:
(Please tick or highlight)

O Mr Lam

O Ms Ziaziaris

O Mr Ireland

Student Number:

(To be used by the exam markers only.)

Question	1-5	6	7	8	Total	Percent
Mark	/5	/16	/15	/14	/50	/100

Section I: Objective Response

Mark your answers on the multiple choice box on the opposite page. **Marks**

1 Suppose $z = 1+3i$ and $\omega = 2-i$. Find $z\overline{\omega}$ in $x+iy$ form: 1

(A) $5-5i$ (B) $5+5i$ (C) $-1+7i$ (D) $5+7i$

2 What is $-\sqrt{3}+i$ expressed in modulus-argument form? 1

(A) $\sqrt{2}\,(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$ (B) $2\,(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

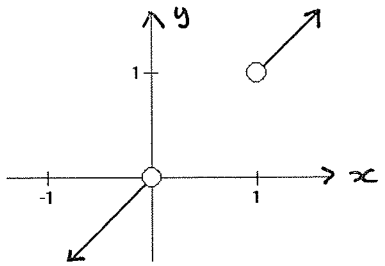
(C) $\sqrt{2}\,(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$ (D) $2\,(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$

3 Given $(2i+1)$ is a root of the equation $x^3-4x^2+9x-10=0$ then another root is: 1

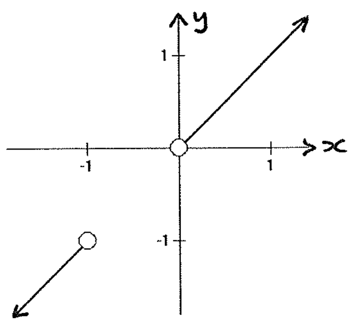
(A) 2 (B) 5 (C) $2i-1$ (D) 10

4 Which of the following represents the locus $\arg z = \arg(z-[1+i])$? 1

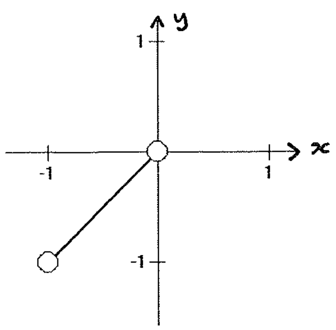
(A)



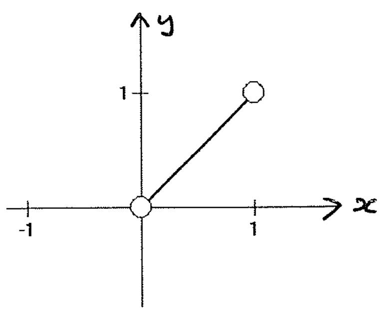
(B)



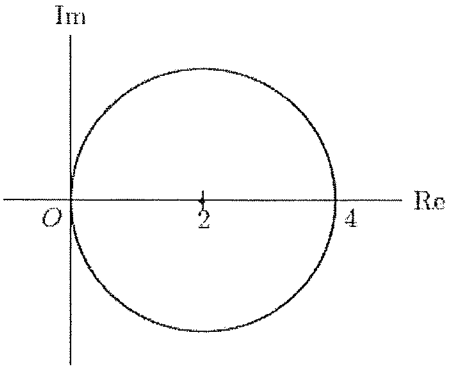
(C)



(D)



5 Which of the following is the equation of the circle shown below? 1



(A) $(z+2)(\bar{z}+2)=4$

(B) $(z-2)(\bar{z}-2)=4$

(C) $(z+2i)(\bar{z}-2i)=4$

(D) $(z+2)(\bar{z}-2)=4$

Answer grid for Section I

Mark answers to Section I by fully blackening the correct circle

1	Ⓐ	Ⓑ	Ⓒ	Ⓓ
2	Ⓐ	Ⓑ	Ⓒ	Ⓓ
3	Ⓐ	Ⓑ	Ⓒ	Ⓓ
4	Ⓐ	Ⓑ	Ⓒ	Ⓓ
5	Ⓐ	Ⓑ	Ⓒ	Ⓓ

End of Section I.

Examination continues overleaf.

Section II: Short Answer

Question 6 (16 marks) Commence a NEW page **Mark**

(a) (i) Simplify $(-2i)^3$. 1

(ii) Hence show carefully on an Argand diagram all complex numbers z such that $z^3 = 8i$. 2

(iii) Express the three cube roots of $8i$ in the form $x+iy$. 2

(b) Let ω be one of the non-real cube roots of 1.

(i) Show that $1+\omega+\omega^2 = 0$ 1

(ii) Evaluate $(-1+\omega+\omega^2)(1-\omega+\omega^2)(1+\omega-\omega^2)$ 2

(iii) Evaluate $(1+\omega)(1+\omega^2)(1+\omega^5)(1+\omega^8)(1+\omega^{11})$ 2

(c) Simplify $\dfrac{\cos 3\theta + i\sin 3\theta}{\cos 2\theta - i\sin 2\theta}$ 2

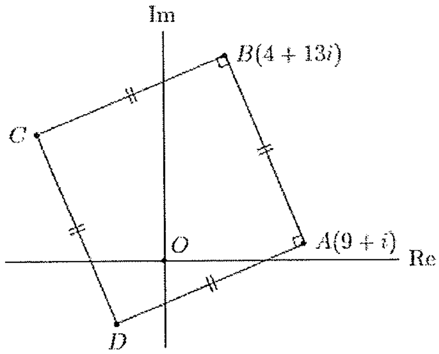
(d) (i) On a single Argand diagram, sketch the locus where $|z-3i| \le 2$ and $\arg(z+1) \le \dfrac{\pi}{4}$ apply. 2

(ii) Find the value of $\arg z$ where $\arg z$ is a minimum. 2

(You may leave answer in exact form)

Question 7 (15 marks) Commence a NEW page **Mark**

(a) In the following diagram, $ABCD$ is a square in the complex plane. Vertices A and B represent the complex numbers $9+i$ and $4+13i$ respectively.



Find the complex numbers represented by:

(i) The vector $\overrightarrow{AB}$ 1

(ii) The vertex D 2

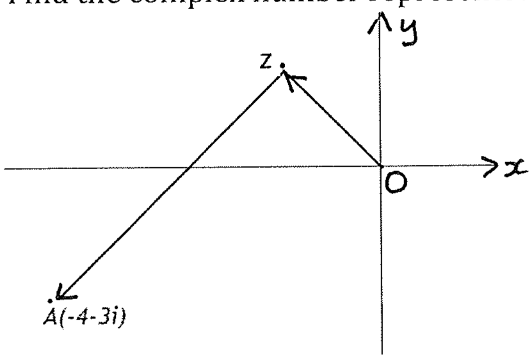
(b) It can be shown $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ and $\sin 3\theta = 3\cos^2\theta\,\sin\theta - \sin^3\theta$.

(i) Deduce that $\tan 3\theta = \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$ 1

(ii) Hence find the roots of $x^3 - 3x^2 - 3x + 1 = 0$ 3

(iii) Without using a calculator, evaluate $\tan\dfrac{\pi}{12} + \tan\dfrac{5\pi}{12}$. 2

(c) The point A represents the complex number $-4-3i$. $\angle OZA = 90°$ and $ZA = 2 \times OZ$. Find the complex number represented by the point Z. 3



(d) Sketch the locus of z if $\dfrac{z-1}{z-2i}$ is purely imaginary. 3

Question 8 (14 marks) Commence a NEW page **Mark**

(a) It is given that $5+6i$ is a zero of $P(x)=2x^3-19x^2+112x+d$, where d is real.

(i) What are the other two zeroes of $P(x)$? 2

(ii) Find the value of d. 2

(b) When $P(x)=x^4+ax^2+bx$ is divided by x^2+1 the remainder is $x+2$.

Find the values of a and b. 2

(c)

(i) Suppose that the polynomial $P(x)$ has a double root at $x=\alpha$.

Prove that $P'(x)$ also has a root at $x=\alpha$. 2

(ii) The polynomial $P(x)=x^4+ax^2+bx+36$ has a double root at $x=2$.

Find the values of a and b. 2

(d) If the equation $x^3-3x^2-x+2=0$ has roots α,β,γ then find the polynomial equation with roots $2\alpha+\beta+\gamma$, $\alpha+2\beta+\gamma$, and $\alpha+\beta+2\gamma$. 3

(e) How many positive integers n are there such that $n+3$ divides n^2+7 without a remainder? 1

END OF EXAM

Q1 - C ✓

Q2 - D ✓

Q3 - A ✓

Q4 - A ✓

Q5 - B ✓

5

[Reasoning for Section 1 :

(1) $z\bar{w} = (1+3i)(2+i)$
$= 2+i+6i-3$
$= -1+7i \quad \therefore (C)$

(2) $-\sqrt{3}+i = 2\left(-\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}\right)$
$= 2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right) \quad \therefore (D)$

(3) Let roots be $1+2i,\ 1-2i,\ \alpha$
Sum of roots is $4 \quad \therefore 2+\alpha=4$
$\therefore \alpha = 2 \quad \therefore (A)$

(4) $\arg(z-z_1)=\arg(z-z_2)$ is the locus of all points dividing z_1 & z_2 externally.
$\therefore \arg(z-0)=\arg(z-(1+i))$ is (A)

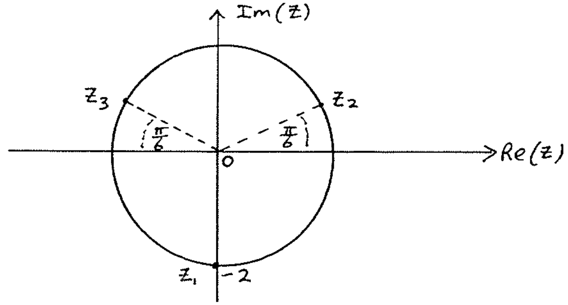
(5) Circle is $(x-2)^2+y^2=4$
i.e. $(x^2+y^2)-2(2x)+4=4$
i.e. $(z-2)(\bar{z}-2)=4 \quad \therefore (B)$]

2014 Extension II – Task 1 Suggested Solutions

$\boxed{Q6}$

(a) (i) $(-2i)^3 = -8i^3$
$= 8i$ ✓

(ii)



✓✓

[One solution, by (i), is at $-2i$, and the other two lie on the circle radius 2, centre (0,0), $\frac{2\pi}{3}$ radians apart.]

(iii) From (ii), $z_1 = 0-2i$
$z_2 = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$
$\therefore z_2 = \sqrt{3}+i$
$z_3 = 2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$
$\therefore z_2 = -\sqrt{3}+i$ ✓✓

(ALT: candidates could use De Moivre $\rightarrow z = 2\left[\text{cis}\left(\frac{2n\pi}{3}+\frac{\pi}{6}\right)\right]$, etc.

or Sum of 2 cubes $\rightarrow z^3+8i^3=0$, etc.)

Q6 – continued

(b) (i) we have $\omega^3 = 1$

$\therefore \omega^3 - 1 = 0$

$\therefore (\omega - 1)(\omega^2 + \omega + 1) = 0$

since ω is non-real, $\therefore \omega^2 + \omega + 1 = 0$. ✓

(ii) $(-1 + \omega + \omega^2)(1 - \omega + \omega^2)(1 + \omega - \omega^2)$

$= (-2)(-2\omega)(-2\omega^2)$ ✓

$= -8\omega^3$

$= -8$ ✓

(iii) $(1+\omega)(1+\omega^2)(1+\omega^5)(1+\omega^8)(1+\omega^{11})$

$= (1+\omega)(1+\omega^2)(1+\omega^2)(1+\omega^2)(1+\omega^2)$ ✓

$= (-\omega^2)(-\omega)(-\omega)(-\omega)(-\omega)$

$= -\omega^6$

$= -1$ ✓

(c) $\dfrac{\cos 3\theta + i \sin 3\theta}{\cos 2\theta - i \sin 2\theta} = \dfrac{\cos 3\theta + i \sin 3\theta}{\cos(-2\theta) + i \sin(-2\theta)}$

$= \dfrac{(\cos\theta + i\sin\theta)^3}{(\cos\theta + i\sin\theta)^{-2}}$

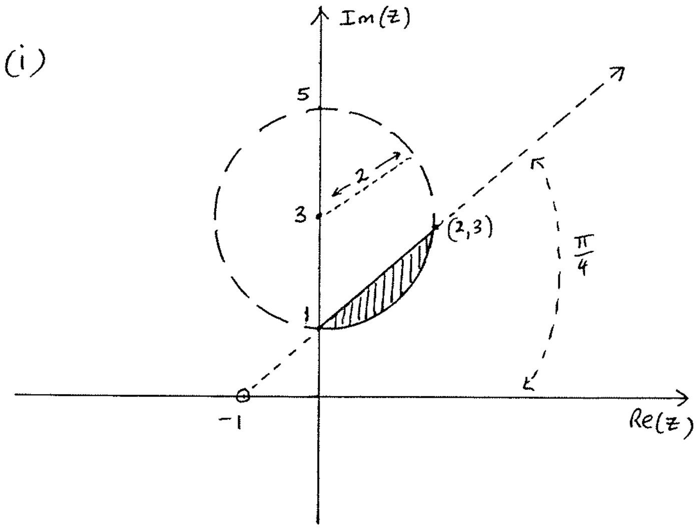
$= (\cos\theta + i\sin\theta)^5$

$= \cos 5\theta + i \sin 5\theta$. ✓✓

2014 Extension II – Task 1 Suggested Solutions

Q6 – continued

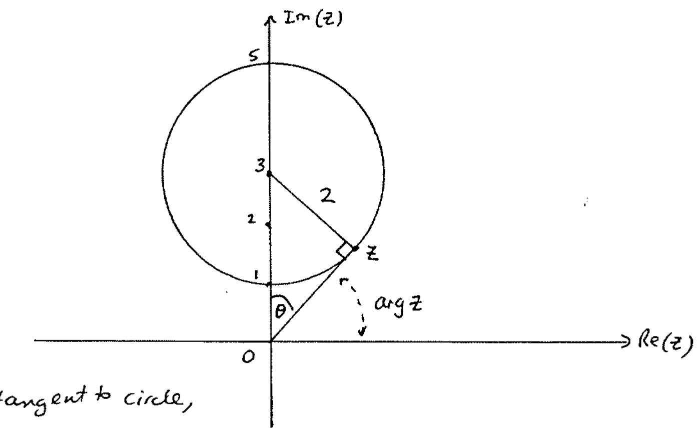
(d) (i)



✓ correctly displays one locus

✓ correctly shows both loci with shading

(ii)



OZ is tangent to circle,

$\therefore \sin\theta = \frac{2}{3}$ ✓ progress towards solution

$\therefore \arg z = \frac{\pi}{2} - \sin^{-1}\left(\frac{2}{3}\right)$ ✓ correct answer

$(\doteqdot 0.841\,069)$.

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Q7

(a) (i) $\overrightarrow{AB}$ represents $-5+12i$ ✓ correct answer

(ii) $\overrightarrow{AD}$ is $\overrightarrow{AB}$ rotated 90° counter-clockwise.

$\therefore \overrightarrow{AD}$ represents $i(-5+12i)$

$= -12 - 5i$ ✓ vector $\overrightarrow{AD}$

$\therefore$ vertex D represents $-3-4i$. ✓ correct answer

(b) (i)
$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$$
$$= \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$
$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \quad \left(\text{Dividing by } \cos^3\theta\right)$$
$$= \frac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}, \text{ as required.}$$
✓ shows all working

(ii) Let $x = \tan\theta$.

The equation becomes:

$\tan^3\theta - 3\tan^2\theta - 3\tan\theta + 1 = 0$

$\therefore \tan^3\theta - 3\tan\theta = 3\tan^2\theta - 1$

$\therefore \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1} = 1$

$\therefore \tan 3\theta = 1$ (from (i)) ✓ Correctly applies part (i)

$3\theta = \frac{\pi}{4} + n\pi$ (n an integer)

$\theta = \frac{4n+1}{12}\cdot\pi$ ✓ Correct θ's.

Q7 – continued

(b) (ii) (continued)

$$\begin{cases} n=0 \rightarrow \theta = \frac{\pi}{12} \\ n=1 \rightarrow \theta = \frac{5\pi}{12} \\ n=2 \rightarrow \theta = \frac{9\pi}{12} = \frac{3\pi}{4} \end{cases}$$

[Note: algebraic solutions giving -1, 2 ± 3 receive 2 marks].

So $x = \tan\frac{\pi}{12}$, $\tan\frac{5\pi}{12}$, $\tan\frac{3\pi}{4} = -1$. ✓ Gives 3 correct roots

(iii) by sum of roots,

$\tan\frac{\pi}{12} + \tan\frac{5\pi}{12} - 1 = 3$ ✓ attempts to use sum of roots

$\therefore \tan\frac{\pi}{12} + \tan\frac{5\pi}{12} = 4$ ✓ correct answer

(c) $\overrightarrow{OZ} + \overrightarrow{ZA} = \overrightarrow{OA}$

Let $z = x+iy$

$\therefore (x+iy) + 2i(x+iy) = -4-3i$ ✓ Correct start

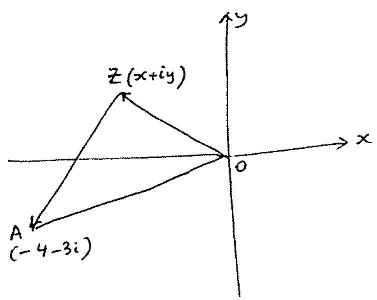
$\therefore x - 2y + i(2x+y) = -4-3i$

$\therefore$ Equating real & imaginary parts,

$$\begin{cases} x - 2y = -4 \\ 2x + y = -3 \end{cases}$$ ✓ correct working

Solving simultaneously,

$x = -2$
$y = 1$



$\therefore Z = -2+i$. ✓ correct answer

[see over for alternate approaches →]

ALTERNATIVE SOLUTIONS to 7(c):

[1] $\vec{OZ} + \vec{ZA} = \vec{OA}$

But $\vec{ZA}$ is $\vec{OZ}$ rotated $\frac{\pi}{2}$ counterclockwise and stretched by a factor of 2.

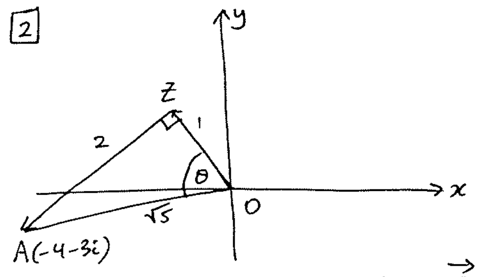
$\therefore z + 2i \cdot z = -4-3i$

$z(1+2i) = -4-3i$

$z = \frac{-4-3i}{1+2i} \times \frac{1-2i}{1-2i}$

$= \frac{-4+8i-3i-6}{5}$ $\quad \therefore z = -2+i$.

[2]



$\vec{OA} = \sqrt{5} \cdot \text{cis}\,\theta \cdot \vec{OZ}$

$= \sqrt{5}\ \text{cis}(\tan^{-1} 2) \cdot \vec{OZ}$

$= \sqrt{5}\left[\frac{1}{\sqrt{5}} + i\frac{2}{\sqrt{5}}\right] \cdot \vec{OZ}$

$= (1+2i) \cdot \vec{OZ}$

$\therefore z = \frac{-4-3i}{1+2i} = -2+i$

[3] $\vec{OZ} = \frac{1}{\sqrt{5}} \cdot \text{cis}(-\theta) \cdot \vec{OA}$

clockwise rotation here

$= \frac{1}{\sqrt{5}}\left(\frac{1}{\sqrt{5}} - i \cdot \frac{2}{\sqrt{5}}\right) \cdot (-4-3i)$

$= \left(\frac{1}{5} - i \cdot \frac{2}{5}\right)(-4-3i)$

$= -2+i$. #

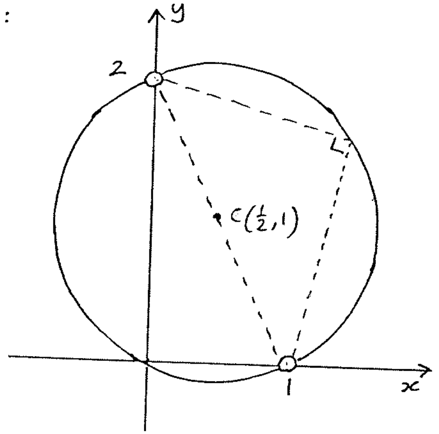
Q7 – continued

(d) $\frac{z-1}{z-2i}$ is purely imaginary.

$\therefore \arg\left(\frac{z-1}{z-2i}\right) = \pm\frac{\pi}{2}$.

This is a circle on diameter AB, where $A = (1,0)$ and $B = (0,2)$, excluding the points A and B.

Hence:



Needs:
- Correct circle locus, through (0,2), (1,0) and (0,0).
- open circle at (0,2)
- centre shown

✓✓✓

ALT: Let $z = x+iy$

$\therefore \frac{z-1}{z-2i} = \frac{(x-1)+iy}{x+i(y-2)} \times \frac{x-i(y-2)}{x-i(y-2)}$ $\quad (z \neq 2i)$

$= \frac{x(x-1)+y(y-2)+i[xy-(x-1)(y-2)]}{x^2+(y-2)^2}$.

Now, Re part $= 0$, $\therefore x(x-1)+y(y-2) = 0$ ✓ first mark here.

$\therefore \left(x-\frac{1}{2}\right)^2 + (y-1)^2 = \frac{5}{4}$ ✓ completes square to get circle

which is a circle, centre $\left(\frac{1}{2},1\right)$, radius $\frac{\sqrt{5}}{2}$, excluding (0,2), as shown above. ✓ Correct sketch (as above)

Note: texts differ on whether (0,0) is regarded as 'purely imaginary', as it lies on both Re and Im axes. See Patel p.114, Lee p.52. Hence difference in the two approaches at (1,0), & no penalty either way.

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Q8

(a) $P(x) = 2x^3 - 19x^2 + 112x + d$, ($d$ real).

(i) Since coefficients real, another zero is $5-6i$. ✓

Let 3rd coefficient be α.

Sum of coefficients $\Rightarrow (5+6i)+(5-6i)+\alpha = \frac{19}{2}$

$\therefore \alpha = -\frac{1}{2}$ ✓

Thus other zeroes are $5-6i, -\frac{1}{2}$.

(ii) product of zeroes $= -\frac{d}{2}$

$\therefore (5+6i)(5-6i)(-\frac{1}{2}) = -\frac{d}{2}$ ✓

$\therefore d = 5^2+6^2 = 61$

$\therefore d = 61$ ✓

(b) $P(x) = x^4 + ax^2 + bx = (x^2+1)\cdot Q(x) + x+2$

Sub. in $x = i \Rightarrow$

$\therefore i^4 + a\cdot i^2 + bi = 0 + i + 2$ ✓

$\therefore (1-a) + bi = 2 + i$

Equate real & imaginary parts,

$1-a = 2$
$b = 1$

$\therefore a = -1$
$b = 1$ ✓

2014 Extension II – Task 1 Suggested Solutions

Q8 – continued

(c)

(i) If $P(x)$ has double root at $x = \alpha$,

then $P(x) = (x-\alpha)^2 \cdot Q(x)$, where $Q(\alpha) \neq 0$

$\therefore P'(x) = Q(x)\cdot 2(x-\alpha) + (x-\alpha)^2 \cdot Q'(x)$

$= (x-\alpha)\left[2\cdot Q(x) + (x-\alpha)\cdot Q'(x)\right]$

so $P'(\alpha) = 0\left[2\cdot Q(\alpha) + 0\cdot Q'(\alpha)\right]$

$= 0$

$\therefore x = \alpha$ is also a root of $P'(x)$. ✓✓

(ii) $P(x) = x^4 + ax^2 + bx + 36$

$\therefore P'(x) = 4x^3 + 2ax + b$

$P(2) = P'(2) = 0$ as $x = 2$ is a double root of $P(x)$.

$\therefore 16 + 4a + 2b + 36 = 0$

$\therefore 2a + b = -26$ ---- (1)

and $32 + 4a + b = 0$

$\therefore 4a + b = -32$ ---- (2)

(2) − (1) → $2a = -6$

$\therefore a = -3$
$b = -20$ ✓✓

$\boxed{Q8}$ continued

(d) $\alpha+\beta+\gamma = 3$

$\therefore$ we want roots $\alpha+3,\ \beta+3,\ \gamma+3$. ✓

$\therefore$ equation is

$(x-3)^3 - 3(x-3)^2 - (x-3) + 2 = 0$ ✓

$\therefore x^3 - 9x^2 + 27x - 27 - 3x^2 + 18x - 27 - x + 3 + 2 = 0$

$\therefore \quad x^3 - 12x^2 + 44x - 49 = 0$ ✓

(e)
$$\begin{array}{r|l} & n \quad -3 \\ \hline n+3 & n^2 + 0n + 7 \\ & n^2 + 3n \\ \hline & \quad -3n + 7 \\ & \quad -3n - 9 \\ \hline & \qquad 16 \end{array}$$

$\therefore \quad n^2 + 7 = (n+3)(n-3) + 16$

So, for $n+3$ to divide n^2+7 with no remainder, we must have $n+3$ divides 16.

Since $n>0$, $\therefore$ $n+3 = 4, 8,$ or 16

$n = 1, 5,$ or 13

$\therefore$ 3 integers. ✓

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