NORTH SYDNEY BOYS HIGH SCHOOL

MATHEMATICS (EXTENSION 2)

2015 HSC Course Assessment Task 1

Friday Dec 5, 2014

General instructions

- Working time – 60 minutes. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 3)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 12M4A – Mr Lam ◯ 12M4B – Mr Ireland ◯ 12M4C – Mr Lin

Marker's use only.

QUESTION	1-5	6	7	8	Total	%
MARKS	$\overline{5}$	$\overline{14}$	$\overline{11}$	$\overline{15}$	$\overline{45}$	$\overline{100}$

Section I

5 marks
Attempt Question 1 to 5

Mark your answers on the answer sheet provided.

Questions **Marks**

1. If $z = \sqrt{2}\left(\cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)\right)$ then z^9 is equal to: **1**

(A) $z = 16\sqrt{2}\left(\cos\left(\frac{36\pi}{5}\right) + i\sin\left(\frac{36\pi}{5}\right)\right)$

(B) $z = 16\sqrt{2}\left(\cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right)\right)$

(C) $z = 16\sqrt{2}\left(\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right)$

(D) $z = 9\sqrt{2}\left(\cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right)\right)$

2. If $P(z) = z^3 - 2z^2 + 4z - 8, z \in \mathbb{C}$, then a linear factor of $P(z)$ is **1**

(A) 2

(B) $2i$

(C) $z + 2$

(D) $z + 2i$

3. The distance between the two points z and $-\overline{z}$ in the complex plane is given by **1**

(A) $2\,\mathrm{Re}\,(z)$

(B) $2\,\mathrm{Im}\,(z)$

(C) $|z|$

(D) $2\,\mathrm{Re}\,(z) + 2\,\mathrm{Im}\,(z)$

4. $P(z)$ is a polynomial in z of degree 4 with real coefficients. Which of the following statements **must** be **false**. **1**

(A) $P(z) = 0$ has no real roots.

(B) $P(z) = 0$ has one real root and three non-real roots.

(C) $P(z) = 0$ has two real roots and two non-real roots

(D) $P(z) = 0$ has four real roots

5. Given that $(1+i)^n = ai$, where a is a non-zero real constant, then $(1+i)^{2n+2}$ simplifies to **1**

(A) a^4

(B) $2a^2 i$

(C) $1 + a^2 i$

(D) $-2a^2 i$

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

1 – Ⓐ Ⓑ Ⓒ Ⓓ
2 – Ⓐ Ⓑ Ⓒ Ⓓ
3 – Ⓐ Ⓑ Ⓒ Ⓓ
4 – Ⓐ Ⓑ Ⓒ Ⓓ
5 – Ⓐ Ⓑ Ⓒ Ⓓ

Examination continues overleaf...

Section II

40 marks
Attempt Questions 6 to 8

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (14 Marks) Commence a NEW page. **Marks**

(a) Express $(2-3i)^2$ in the form $a+ib$, where a and b are real **2**

(b) z is the complex number $1+i$ Write the following in modulus-argument form

i. z **2**

ii. iz **1**

(c) Sketch on the Argand diagram the locus $|z-1| = |z+1-2i|$ **2**

(d) On an Argand diagram, graph the intersection of the regions defined by **4**

$$z\overline{z} \geq 9, \;\; z+\overline{z} \leq 8 \text{ and } 0 < \arg z < \frac{\pi}{4}$$

(e) Let ω be one of the non-real cube roots of 1

i. Show that $1+\omega+\omega^2 = 0$ **1**

ii. Hence or otherwise, prove that $(1+\omega-\omega^2)^3 - (1-\omega+\omega^2)^3 = 0$. **2**

Question 7 (11 Marks) Commence a NEW page. **Marks**

(a) Find constants A , B, and C so that **2**

$$\frac{1}{1+x+x^2+x^3} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

(b) Find the integers m and n such that $(x+1)^2$ is a factor of the polynomial **3**

$$P(x) = x^5 + 2x^2 + mx + n$$

(c) If α, β, γ are the roots of $x^3 + mx + n = 0$, form the equation whose roots are **3**

$\dfrac{1}{\alpha+\beta}, \dfrac{1}{\beta+\gamma}, \dfrac{1}{\gamma+\alpha}$

(d) The equation $x^3 + px^2 + qx + r = 0$ has one root equal to the sum of the other two. Show that $p^3 - 4pq + 8r = 0$ **3**

Examination continues overleaf...

Question 8 (15 Marks) Commence a NEW page. **Marks**

(a) The four complex numbers z_1, z_2, z_3, z_4 are represented on the complex plane by the points A, B, C, D respectively. **4**

Given that

$$z_1 - z_2 + z_3 - z_4 = 0 \quad \text{and} \quad z_1 - iz_2 - z_3 + iz_4 = 0$$

use vectors to determine the possible shape(s) for the quadrilateral $ABCD$. Show all reasoning.

(b) Consider the polynomial equation: $z^5 - i = 0$

i. Find all the roots of $z^5 - i = 0$. You may leave the roots in the form of $\operatorname{cis}\theta$. **2**

ii. Hence show that **3**

$$(z - i)\left[z^2 - \left(2i\sin\frac{\pi}{10}\right)z - 1\right]\left[z^2 + \left(2i\sin\frac{3\pi}{10}\right)z - 1\right] = 0$$

iii. Hence or otherwise deduce that $\sin\dfrac{3\pi}{10} - \sin\dfrac{\pi}{10} = \dfrac{1}{2}$ **2**

(c) If $\dfrac{z - 1 + i}{z + 1 - i} = ki$, where $k \in \mathbb{R}$ **4**

By drawing a diagram or otherwise, find the value of $|z|$. Show all reasoning.

End of paper.

Multiple Choice

1. C 2. D 3. A 4. B 5. D

Question 6

a) $(2-3i)^2 = 4 + 2(-3i)\times 2 + 9i^2$

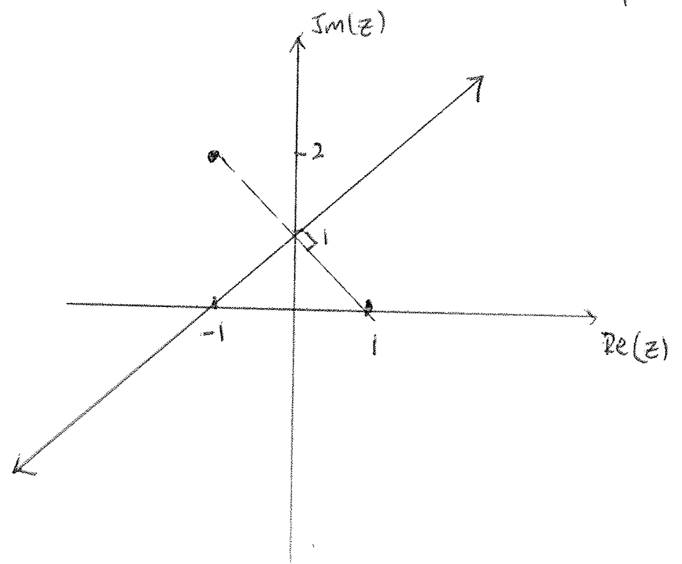
$= -5 - 12i$

b) (i) $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

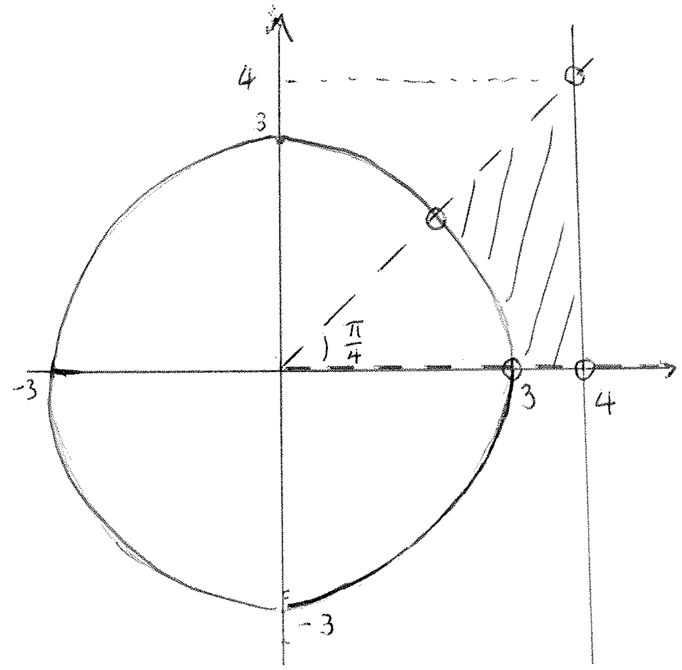
(ii) $iz = \sqrt{2}\left(\cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{2}\right)\right)$

$= \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

c)



d)



$z\bar{z} \geq 9 \Rightarrow x^2+y^2 \geq 9$

$z+\bar{z} \leq 8 \Rightarrow 2\,\mathrm{Re}(z) \leq 8$

$\mathrm{Re}(z) \leq 4$

$x \leq 4$

e) i) Since ω is a non real cube root of $z^3-1=0$

and $(z-1)(1+z+z^2)=0$ upon factorising

then $1+\omega+\omega^2=0$ as

$\omega - 1 \neq 0$

ii) As $1+\omega+\omega^2=0$

$\Rightarrow 1+\omega = -\omega^2$

and $1+\omega^2 = -\omega$

LHS $= (1+\omega-\omega^2)^3 - (1-\omega+\omega^2)^3$

$= (-\omega^2-\omega^2)^3 - (-\omega-\omega)^3$

$= (-2\omega^2)^3 - (-2\omega)^3$

$= -8(\omega^3)^2 - (-8\omega^3)$

$= -8 + 8$ (as $\omega^3 = 1$)

$= 0$

$=$ RHS

Question 7

a) $\dfrac{1}{1+x+x^2+x^3} \equiv \dfrac{A}{1+x} + \dfrac{Bx+C}{1+x^2}$

$\Rightarrow A(1+x^2) + (1+x)(Bx+c) \equiv 1$

when $x=-1$

$2A = 1$

$A = \frac{1}{2}$

when $x=0$

$A + C = 1$

$C = \frac{1}{2}$

equating coefficient of x^2

$A + B = 0$

$B = -\frac{1}{2}$.

b) $P'(x) = 5x^4 + 4x + m$

Because there is a double root at $x=-1$

$\therefore P'(-1) = 0$

$P'(-1) = 5 - 4 + m = 0$

$\therefore m = -1$

But $P(-1) = 0$

$\therefore P(-1) = (-1)^5 + 2(-1)^2 - (-1) + n = 0$

$\therefore 2 + n = 0$

$n = -2$

c) $x^3 + mx + n = 0$

Sum of the roots

$\alpha + \beta + \gamma = 0$

$\therefore \alpha + \beta = -\gamma$

$\dfrac{1}{\alpha+\beta} = -\dfrac{1}{\gamma}$

Similarly

$\dfrac{1}{\beta+\gamma} = -\dfrac{1}{\alpha}$

and $\dfrac{1}{\alpha+\gamma} = -\dfrac{1}{\beta}$

Substitute x with $-\frac{1}{x}$

$\left(-\frac{1}{x}\right)^3 + m\left(-\frac{1}{x}\right) + n = 0$

$\therefore nx^3 - mx^2 - 1 = 0$

d) $x^3 + px^2 + qx + r = 0$

roots are α, β, γ

$\alpha = \beta + \gamma$

Sum of the roots: $2\beta + 2\gamma = -p$

$\therefore \beta + \gamma = -\frac{p}{2}$

Now $\beta + \gamma$ is a root $\therefore -\frac{p}{2}$ is a root

Sub $\frac{-p}{2}$ in $x^3 + px^2 + qx + r = 0$

$\left(\frac{-p}{2}\right)^3 + p\left(\frac{-p}{2}\right)^2 + q\left(\frac{-p}{2}\right) + r = 0$

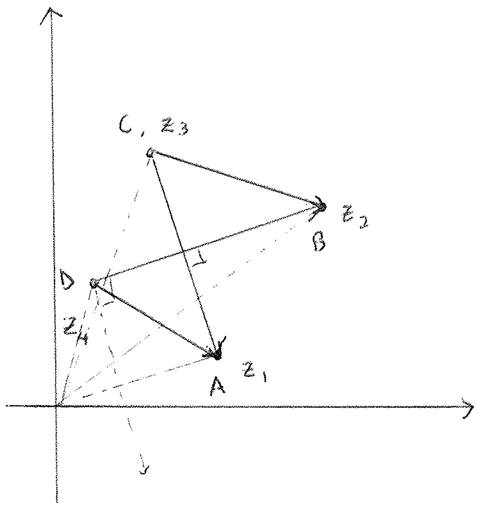
$-\frac{p^3}{8} + \frac{p^3}{4} - \frac{pq}{2} + r = 0$

multiply through by 8

$p^3 - 4pq + 8r = 0$

Question 8

a)



If $z_1 - z_2 + z_3 - z_4 = 0$

$z_1 - z_4 = z_2 - z_3$

$\vec{CB} = \vec{DA}$

$\Rightarrow$ CB = AD

CB ∥ AD

∴ ABCD is a parallelogram

(one pair of opposite sides equal in length and parallel)

Now if $z_1 - iz_2 - z_3 + iz_4 = 0$

$z_1 - z_3 = i(z_2 - z_4)$

$\Rightarrow$ AC is a image of BD with rotation of 90°

i.e. AC ⊥ BD and AC = BD

∴ diagonals are perpendicular and equal in length

∴ ABCD can only be a square.

b) i) $z^5 - i = 0 \Rightarrow z^5 = i = cis(\frac{\pi}{2} + 2k\pi)$, $k \in \mathbb{Z}$

$z = i^{\frac{1}{5}} = (cis(\frac{\pi}{2} + 2k\pi))^{\frac{1}{5}}$

∴ using de Moivre's Theorem

$z = cis\left(\frac{\pi + 4k\pi}{10}\right)$

For 5 roots, $k = -2, -1, 0, 1, 2$

$z_1 = cis\left(\frac{-7\pi}{10}\right)$ $z_4 = cis\left(\frac{5\pi}{10}\right) = i$

$z_2 = cis\left(-\frac{3\pi}{10}\right)$ $z_5 = cis\left(\frac{9\pi}{10}\right)$

$z_3 = cis\left(\frac{\pi}{10}\right)$

ii) Now $z^5 - i = (z - i)(z - cis\frac{\pi}{10})(z - cis\frac{9\pi}{10})(z - cis(-\frac{3\pi}{10}))(z - cis(-\frac{7\pi}{10}))$

For $(z - cis\frac{\pi}{10})(z - cis\frac{9\pi}{10})$

$= z^2 - z\,cis\frac{9\pi}{10} - z\,cis\frac{\pi}{10} + cis\frac{\pi}{10}\,cis\frac{9\pi}{10}$

$= z^2 - z\left[\cos\frac{9\pi}{10} + i\sin\frac{9\pi}{10} + \cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right] + cis\,\pi$

$= z^2 - z\left[-\cos\left(\frac{\pi}{10}\right) + i\sin\frac{\pi}{10} + \cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right] - 1$

as $\cos(\pi - x) = -\cos x$

and $\sin(\pi - x) = \sin x$

$\cos\pi + i\sin\pi = -1$

$= z^2 - z\left[2i\sin\frac{\pi}{10}\right] - 1$

similarly for $\left(z - \text{cis}\left(-\frac{3\pi}{10}\right)\right)\left(z - \text{cis}\left(-\frac{7\pi}{10}\right)\right)$

$= z^2 - z\left[\text{cis}\left(-\frac{3\pi}{10}\right) + \text{cis}\left(-\frac{7\pi}{10}\right)\right] + \text{cis}\left(-\frac{3\pi}{10}\right)\text{cis}\left(-\frac{7\pi}{10}\right)$

$= z^2 - z\left[\cos\left(-\frac{3\pi}{10}\right) + i\sin\left(-\frac{3\pi}{10}\right) + \cos\left(-\frac{7\pi}{10}\right) + i\sin\left(-\frac{7\pi}{10}\right)\right] + \text{cis}(-\pi)$

$= z^2 - z\left[\cos\frac{3\pi}{10} - i\sin\frac{3\pi}{10} + \cos\frac{7\pi}{10} - i\sin\frac{7\pi}{10}\right] - 1$ as $\cos\theta$ is even, $\sin\theta$ is odd

$= z^2 - z\left[\cos\frac{3\pi}{10} - i\sin\frac{3\pi}{10} - \cos\frac{3\pi}{10} - i\sin\frac{3\pi}{10}\right] - 1$

$= z^2 + z\left[2i\sin\frac{3\pi}{10}\right] - 1$

$\therefore z^5 - i = (z - i)\left(z^2 - \left(2i\sin\frac{\pi}{10}\right)z - 1\right)\left(z^2 + \left(2i\sin\frac{3\pi}{10}\right)z - 1\right) = 0$

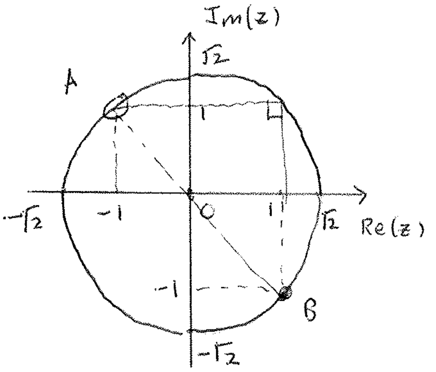
ii) From sum of the roots

$i + \text{cis}\frac{\pi}{10} + \text{cis}\frac{9\pi}{10} + \text{cis}\left(-\frac{3\pi}{10}\right) + \text{cis}\left(-\frac{7\pi}{10}\right) = 0$

From (ii) $2i\sin\frac{\pi}{10} - 2i\sin\frac{3\pi}{10} = -i$

$\sin\frac{3\pi}{10} - \sin\frac{\pi}{10} = \frac{1}{2}$

c) $\arg\left(\frac{z - (1 - i)}{z - (-1 + i)}\right) = \pm\frac{\pi}{2}$



$\therefore$ circle with origin $(0, 0)$ radius $\sqrt{2}$

$\therefore |z| = \sqrt{2}$

Note that $(1, -1)$ is actually part of the solution

- Forms a right angle at circumference
  $\therefore$ AB is the diameter of a circle