SAINT IGNATIUS' COLLEGE

RIVERVIEW

YEAR 12

EXTENSION TWO MATHEMATICS

ASSESSMENT TASK 1

November 2007

Time allowed: one hour

Instructions to students

- All questions may be attempted.
- All necessary working should be shown in every question.
- Marks for each part in a question shown on the paper.
- Full marks may not be awarded for careless or badly arranged work.
- Board approved calculators and templates may be used.
- The answers to the two questions in this paper are to be returned in separate booklets clearly marked QUESTION 1 and QUESTION 2 on the front cover of the booklet.
- **Write your name on the front cover of each booklet.**

Question 1 (24 marks) Use a SEPARATE writing booklet **Marks**

(a) Evaluate:

(i) $|5-2i|$ 1

(ii) $\arg(-3+3i)$ 1

(b) Let $z=2+i$ and $w=1-i$.
Find in the form $x+iy$,

(i) $3z+iw$ 1

(ii) $z\overline{w}$ 2

(iii) $\dfrac{5}{z}$ 2

(c) (i) Find all pairs of integers a and b such that $(a+ib)^2=8+6i$. 2

(ii) Hence solve: $z^2+2z(1+2i)-(11+2i)=0$. 3

(d) z is a complex number. Show that $z+\dfrac{|z|^2}{z}$ is real. 2

Question 1 continues on page 2

(e) (i) If $z = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$, find z^6. 1

(ii) List and also plot on an Argand diagram, all complex numbers that are the solutions of $z^6 = 1$. Answers can be left in modulus-argument form. 2

(f) (i) Express $z_1 = -1 + i$ and $z_2 = 1 - \sqrt{3}\,i$ in modulus-argument form. 4

(ii) Find $z_1 z_2$. 1

(iii) Hence find the exact value of $\sin\dfrac{5\pi}{12}$. 2

Question 2 (14 marks) Use a SEPARATE writing booklet **Marks**

(a) On separate Argand diagrams, sketch the locus of z described by each of the following conditions:

(i) $|z| \leq 3$ and $0 \leq \arg z \leq \frac{\pi}{3}$ 2

(ii) $2|z| = z + \overline{z} + 4$ 3

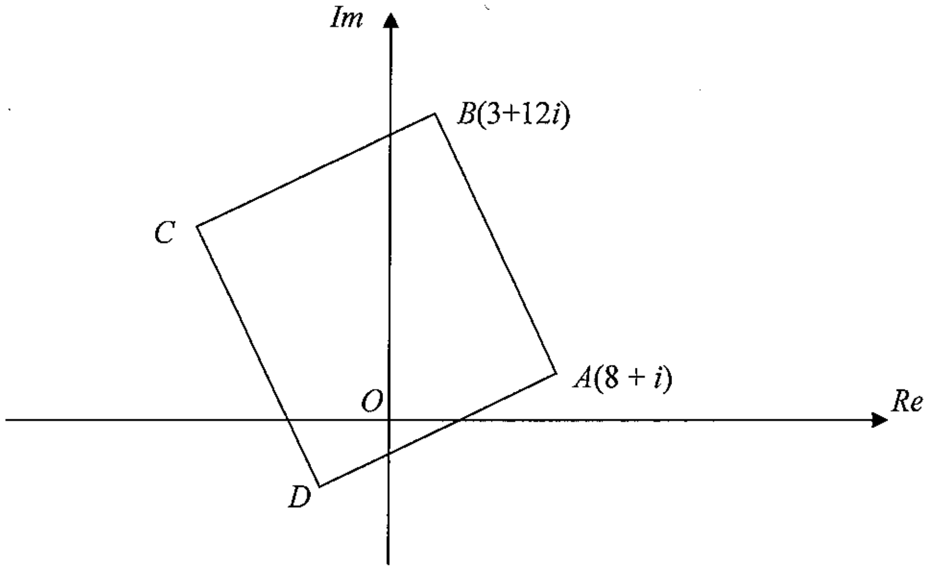
(b) The locus of the complex number z is defined by the equation $\arg(z+1) = \frac{\pi}{4}$.

(i) Sketch the locus of z. 1

(ii) Find the least value of $|z|$. 2

Question 2 continues on page 4

(c)



The diagram above shows a square $ABCD$ in the complex plane. The vertices A and B represent the complex numbers $(8 + i)$ and $(3 + 12i)$ respectively. Find the complex numbers represented by:

(i) the vector AB, 1

(ii) the vertex D. 2

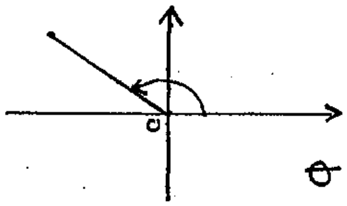
(d) If $w = \dfrac{1+z}{1-z}$ and $|z| = 1$ where w and z are complex numbers, determine the locus of w. 3

End of Assessment task

Question 1

(a) i) $|5-2i| = \sqrt{5^2+2^2}$
$= \sqrt{29}$ ✓

ii)



Let θ be the argument of $-3+3i$.
θ is an obtuse angle
$\tan\theta = -1$
$\theta = \frac{3\pi}{4}$ ✓

b) $z = 2+i, \quad w = 1-i$

i) $3z + iw = 3(2+i) + i(1-i)$
$= 6 + 3i + i + 1$
$= 7 + 4i$ ✓

ii) $z\overline{w} = (2+i)(1+i)$
$= 2 + 3i - 1$ ✓
$= 1 + 3i$ ✓

iii) $\frac{5}{z} = \frac{5}{2+i} \times \frac{2-i}{2-i}$ ✓
$= \frac{10-5i}{4+1}$
$= \frac{10-5i}{5}$
$= 2 - i$ ✓

c) i) $(a+ib)^2 = 8+6i$

$$a^2 + 2abi - b^2 = 8 + 6i$$

$$(a^2 - b^2) + 2abi = 8 + 6i$$

Equating real and imaginary parts,

$$a^2 - b^2 = 8 \quad \ldots\ldots (1)$$

$$2ab = 6 \quad \ldots\ldots (2)$$

Solving (1) and (2) simultaneously,

$$ab = 3$$

$$a = \frac{3}{b}$$

$$\frac{9}{b^2} - b^2 = 8$$

$$9 - b^4 = 8b^2$$

$$b^4 + 8b^2 - 9 = 0$$

$$(b^2+9)(b^2-1) = 0$$

$b^2 = -9$ or $b^2 = 1$

no real sol'n $\qquad b = \pm 1$

when $b = 1$, $a = 3$

when $b = -1$, $a = -3$

$\therefore$ solution is $a = 3, b = 1$ or $a = -3, b = -1$.

ii) $$z = \frac{-2(1+2i) \pm \sqrt{4(1+2i)^2 + 4(11+2i)}}{2}$$

$$= \frac{-2(1+2i) \pm \sqrt{4(1+4i-4) + 44 + 8i}}{2}$$

$$= \frac{-2(1+2i) \pm \sqrt{-12 + 16i + 44 + 8i}}{2}$$

$$= \frac{-2(1+2i) \pm \sqrt{32+24i}}{2}$$

$$= \frac{-2(1+2i) \pm 2\sqrt{8+6i}}{2}$$

$$= -(1+2i) \pm \sqrt{8+6i}$$

From (i) $\sqrt{8+6i} = \pm(3+i)$

$z = -1-2i+3+i$ OR $z = -1-2i-3-i$

$= 2-i$ $\qquad = -4-3i$

d) Let $z = a+ib$ $\qquad |z| = \sqrt{a^2+b^2}$

$$z + \frac{|z|^2}{z} = a+ib + \frac{a^2+b^2}{a+ib}$$

$$= \frac{(a+ib)^2 + a^2+b^2}{a+ib}$$

$$= \frac{a^2+2abi-b^2+a^2+b^2}{a+ib}$$

$$= \frac{2a(a+ib)}{(a+ib)}$$

$= 2a$

which is real.

OR could have used $|z|^2 = z\bar{z}$

$z + \frac{\cancel{z}\bar{z}}{\cancel{z}}$

$z + \bar{z}$ is real

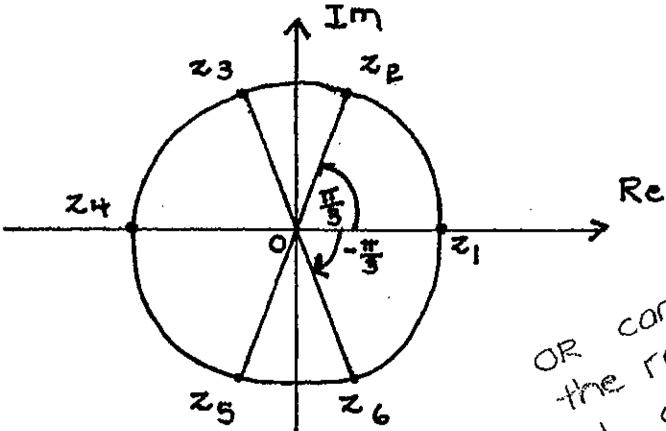
e) i) $z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$$
\begin{aligned}
z^6 &= \cos\left(6\times\frac{\pi}{3}\right) + i\sin\left(6\times\frac{\pi}{3}\right) \\
&= \cos 2\pi + i\sin 2\pi \\
&= 1 + 0i \\
&= 1.
\end{aligned}
$$

ii) We know that $z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ is a solution to $z^6 = 1$. (from part i))

$z = 1$ and $z = -1$ are also solutions.

The 6 solutions to $z^6 = 1$ are evenly spaced around the unit circle.



$z_1 = \cos 0 + i\sin 0$

$z_2 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$z_3 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

OR can use the roots of $z^6 = 1$ are given by $z = \cos\left(\frac{2k\pi}{6}\right) + i\sin\left(\frac{2k\pi}{6}\right)$ for $k = 0, 1, 2, 3, 4, 5$.

$z_4 = \cos\pi + i\sin\pi$

$z_5 = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)$ or $\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$

$z_6 = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$ or $\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)$.

f) i) $z_1 = -1 + i$

$|z_1| = \sqrt{2}$ ✓

$\arg(z_1) = \frac{3\pi}{4}$ ✓

$$z_1 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$z_2 = 1 - \sqrt{3}\,i$

$|z_2| = \sqrt{4}$

$= 2$

($\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ ✓
z_2 lies in 4^{th} quadrant)

$\arg(z_2) = -\frac{\pi}{3}$ ✓

$$z_2 = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

ii) $$z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)\right)$$

$$= 2\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

iii) $z_1 z_2 = (-1+i)(1-\sqrt{3}\,i)$

$= -1 + \sqrt{3}\,i + i + \sqrt{3}$

$= (\sqrt{3}-1) + i(\sqrt{3}+1)$ ✓

$$\therefore (\sqrt{3}-1) + i(\sqrt{3}+1) = 2\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

Equating imaginary parts,

$2\sqrt{2}\sin\frac{5\pi}{12} = \sqrt{3}+1$ ✓

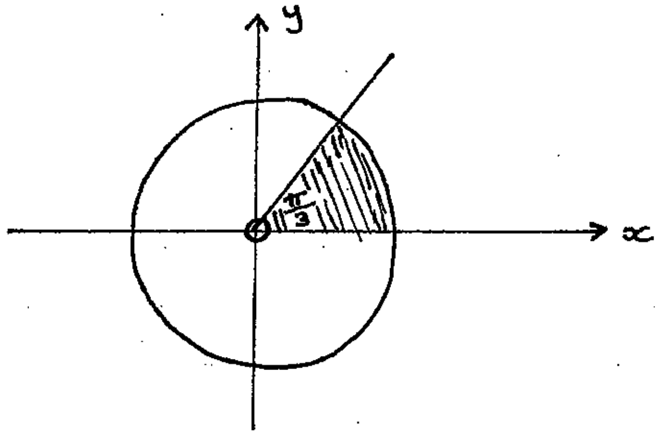
$\sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$ ✓

OR

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

Question 2

a) i)



ii) $2|z| = z + \bar{z} + 4$

let $z = x + iy$

$$2\sqrt{x^2+y^2} = x + iy + x - iy + 4$$

$$2\sqrt{x^2+y^2} = 2x + 4$$

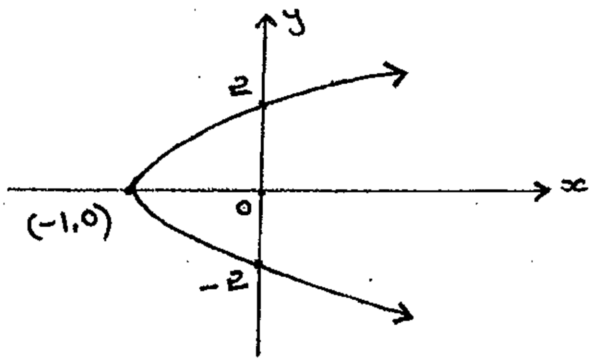
$$\sqrt{x^2+y^2} = x + 2$$

$$x^2 + y^2 = (x+2)^2$$

$$x^2 + y^2 = x^2 + 4x + 4$$

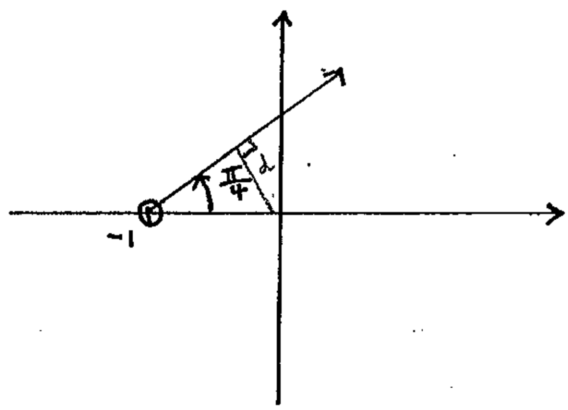
$$y^2 = 4(x+1)$$

$\therefore$ locus is the parabola with vertex $(-1, 0)$ and focus $(0, 0)$.



b) $\arg(z+1) = \frac{\pi}{4}$

i)



ii) Equation of locus of z.

$$y = x + 1 \ , \quad x > -1$$

Least value of $|z|$ is given by the perpendicular distance of the origin to the line $y = x+1$.

$a = 1$
$b = -1$
$c = 1$
$x_1 = 0$
$y_1 = 0$

$$d = \frac{|1|}{\sqrt{1^2+1^2}}$$

$$= \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}$$

or can use trig,

$$\sin\frac{\pi}{4} = \frac{d}{1}$$

$$d = \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}$$

$\therefore$ least value of $|z|$ is $\frac{\sqrt{2}}{2}$ units.

c) i) $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$$8 + i + \overrightarrow{AB} = 3 + 12i$$

$$\overrightarrow{AB} = 3 + 12i - 8 - i$$

$$= -5 + 11i$$

ii) $\overrightarrow{AD}$ is represented by $(-5+11i)\,i$
$= -11-5i$

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$$
$$\overrightarrow{OD} = (8+i) + (-11-5i)$$
$$= -3-4i$$

so D is represented by $(-3-4i)$.

d) $w = \dfrac{1+z}{1-z}$ and $|z| = 1$.

let $z = x + iy$

OR see GJA's solution

$$w = \frac{1+x+iy}{1-(x+iy)}$$
$$= \frac{(1+x)+iy}{(1-x)-iy} \times \frac{(1-x)+iy}{(1-x)+iy}$$

$$= \frac{(1+x)(1-x)+(1+x)iy+(1-x)iy-y^2}{(1-x)^2+y^2}$$

$$= \frac{1-x^2+iy+\cancel{xyi}+iy-\cancel{xyi}-y^2}{1-2x+x^2+y^2}$$
$$= \frac{1-(x^2+y^2)+2yi}{1-2x+x^2+y^2}$$

given that $|z|=1$, then $x^2+y^2=1$

so $w = \dfrac{1-1+2yi}{1-2x+1}$

$$= \frac{2yi}{2(1-x)}$$

$= \dfrac{y}{(1-x)}\,i$ which is purely imaginary
$\therefore$ the locus of w is the y-axis. or the imaginary axis.

$$W = \frac{1+Z}{1-Z}$$

$$(1-Z)W = 1+Z$$

$$W - ZW = 1+Z$$

$$W-1 = ZW + Z$$

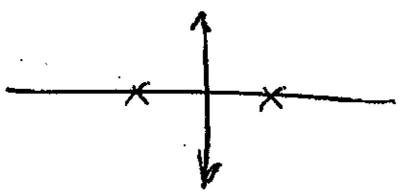
$$W-1 = Z(W+1)$$

$$\frac{W-1}{W+1} = Z$$

$$\frac{|W-1|}{|W+1|} = |Z|$$

$$|W-1| = |W+1|$$

$$|W-1| = |W-(-1)|$$



$\underline{x=0}$