Name: .... **Maths Class:** ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 11

Extension 2 Mathematics

Assessment 1
HSC Course

December, 2014

Time allowed: 70 minutes

General Instructions:

- Questions are not of equal value
- Approved calculators may be used
- All necessary working should be shown
- Begin each question on a new page
- Write using black or blue pen
- Full marks may not be awarded for careless work or illegible writing

QUESTION 1: (12 Marks)

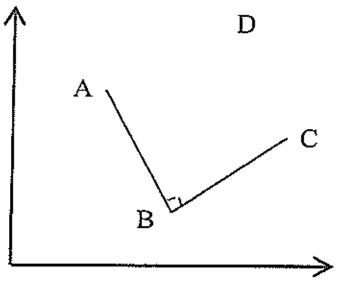
(a) Given that $z = 2 + i$ and $w = 1 - i$ find, in the form $a + ib$ **2**

(i) $z\overline{w}$ (ii) z/w

(b) Shade the area where $|z - 1 - i| < 2$ and $0 < \arg(z - 1 - i) < \pi/4$ hold simultaneously. **3**

(c) In the diagram below, A, B and C are the points representing the complex numbers z_1, z_2, and z_3 respectively.

<ABC is a right angle, and AB = BC



(i) Explain why $(z_1 - z_2)^2 = -(z_3 - z_2)^2$ **2**

(ii) If ABCD is a square, find, in terms of z_1, z_2, and z_3 the complex number represented by the point D **2**

(d) Find the values of x and y if $(x + iy)^2 = 24 + 10i$ **3**

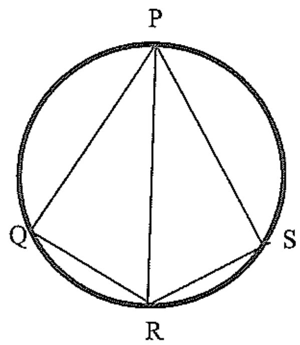
QUESTION 2: (11 Marks)

(a) (i) Show that $z\bar{z} = |z|^2$ **1**

(ii) If $z = 2(\cos\theta + i\sin\theta)$ find $\overline{1-z}$ in terms of θ **1**

(iii) Show that $Re\left(\frac{1}{1-z}\right) = \frac{1-2\cos\theta}{5-4\cos\theta}$ **3**

(b) In the circle below, <QPR = < SPR and <QRP = <SRP.
It is NOT <u>given</u> that PR is a diameter.



Showing all working and reasoning, prove that PR is a diameter. **3**

(c) If $x = 1 + i\sqrt{3}$ find the value of x^{11} in the form $a + ib$ **3**

QUESTION 3: (10 Marks)

(a) (i) If $\alpha = -1 + i$, express α in mod-arg form. **2**

(ii) Show that α satisfies the complex equation $z^4 + 4 = 0$ **2**

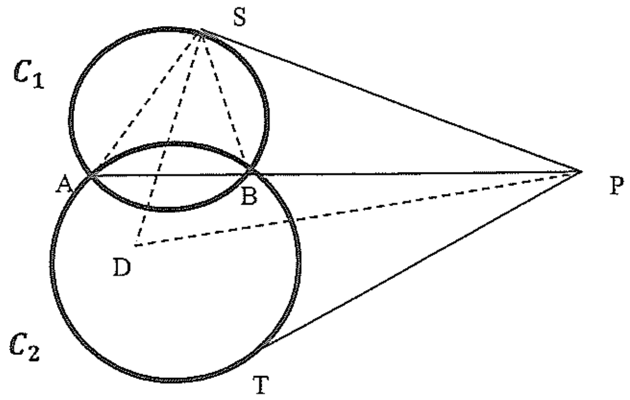
(iii) Hence, or otherwise, factorise $z^4 + 4$ into two Real quadratic factors **2**

(b) If n is an even positive integer, show, without multiplying out, that **4**

$$(1 + x + x^2 + \cdots x^n) \times (1 - x + x^2 - \cdots x^n) = (1 + x^2 + \cdots x^{2n})$$

QUESTION 4: (11 Marks)

(a) Two circles C_1 and C_2 intersect in the points A and B



AB is produced to the point P.

From the point P the tangents PT and PS are drawn as shown.

Redraw the diagram onto your answer sheet **(no marks)**

(i) Prove that $\Delta\, ASP$ is similar to $\Delta\, SBP$ **2**

(ii) Hence prove that $SP^2 = AP \times PB$ **(DO NOT USE THE INTERCEPT THEOREM)** **1**

(iii) Deduce that $PT = PS$ **1**

(iv) The perpendicular from S to SP meets the bisector of $< SPT$ at D. **3**

Prove that DT passes through the centre of the circle C_2

(b) (i) How many terms are there in the geometric series **1**

$2^N + 2^{N-1} + 2^{N-2} \ldots + 2^{1-N} + 2^{-N}$, if N is a positive integer?

(ii) Prove that **2**

$2^N + 2^{N-1} + 2^{N-2} \ldots + 2^{1-N} + 2^{-N} = 2^{N+1} - 2^{-N}$

(iii) As $N \rightarrow \infty$, is there a limiting sum to this series? Justify your answer. **1**

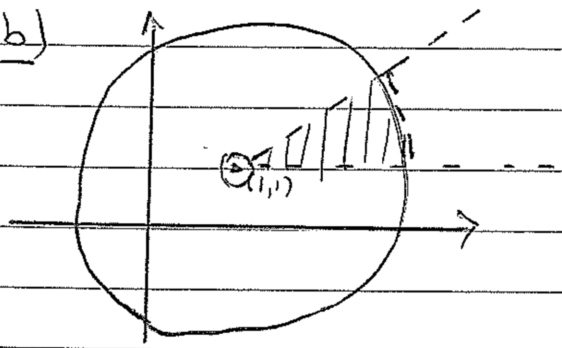
4 UNIT SOLUTIONS

(1) (a) (i) $(2+i)(1+i) = 1+3i$

(ii) $z/w = \dfrac{z\bar{w}}{|w|^2}$

$= \dfrac{1+3i}{2}$ ← from part (i)

(b)



(c) (i) CB is $z_3 - z_2$

AB is $z_1 - z_2$

AB is CB rotated by $90°$

$\therefore z_1 - z_2 = i(z_3 - z_2)$

$\therefore (z_1 - z_2)^2 = -(z_3 - z_2)^2$

(ii) D is $z_1 + (z_3 - z_2)$

(d) $x^2 - y^2 = 24$ and $2xy = 10$

$\Rightarrow y = 5/x$.

$\therefore x^2 - 25/x^2 = 24$

$\therefore x^4 - 24x^2 - 25 = 0$

$(x^2 - 25)(x^2 + 1) = 0$

Taking only the result $x = 5,\ y = 1$

$\therefore \begin{cases} x = 5 \\ y = 1 \end{cases}$ OR $\begin{cases} x = -5 \\ y = -1 \end{cases}$

(2) (i) Let $z = x + iy$, $z\bar{z} = (x+iy)(x-iy)$

$= x^2 + y^2$

$= |z|^2$

(ii) $z = 2\cos\theta + 2i\sin\theta$

$\therefore 1 - z = 1 - 2\cos\theta - 2i\sin\theta$

and $\overline{1-z} = 1 - 2\cos\theta + 2i\sin\theta$.

(iii) $\text{Re}\left(\frac{1}{1-z}\right) = \text{Re}\left[\frac{\overline{1-z}}{|1-z|^2}\right]$

$= \frac{1-2\cos\theta}{(1-2\cos\theta)^2 + 4\sin^2\theta}$

$= \frac{1-2\cos\theta}{5-4\cos\theta}$.

(b) ONE METHOD: Let $\angle QPR = \angle SPR = x$

and $\angle QRP = \angle SRP = y$

$\therefore \angle PQR = 180^\circ - (x+y)$ (angle sum of $\triangle PQR$)

and $\angle PSR = 180 - (x+y)$ (" " " $\triangle PSR$)

ALSO $\angle PQR + \angle PSR = 180^\circ$ (opposite angles in a cyclic quadrilateral)

$\therefore 180^\circ - (x+y) + 180 - (x+y) = 180$

$\Rightarrow x+y = 90^\circ$

$\therefore \angle PQR = 90^\circ$ (angle sum of $\triangle PQR$)

$\therefore$ PR is a diameter (angle in a s/circle is 90°)

(c) $x = 2\,\text{cis}\,\pi/3$

$x^{11} = 2^{11}\,\text{cis}\,11\pi/3$ and $11\pi/3 = -\pi/3$

$\therefore x^{11} = 2^{11}\left[\cos(-\pi/3) + i\sin(-\pi/3)\right]$

$= 2^{11}\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right)$

$= 1024 - 1024\sqrt{3}\,i$

(3) (a) (i) $\alpha = \sqrt{2}\,\text{cis}\,(3\pi/4)$

(ii) $\alpha^4 + 4 = 4\,\text{cis}\,3\pi + 4$

$= -4 + 4$

$= 0$

(iii) If $-1+i$ is a factor, so is $-1-i$

$\therefore$ One factor is $(z+1-i)(z+1+i) = z^2 + 2z + 2$

$\therefore$ the other is $z^2 - 2z + 2$ (inspection)

(b) $S_1 = \frac{1(1-x^{n+1})}{1-x}$ $S_2 = \frac{1-(-x)^{n+1}}{1+x}$

$= \frac{1-x^{n+1}}{1-x}$ $= \frac{1+x^{n+1}}{1+x}$ since n is even

$\therefore$ and $S_3 = \frac{1-(x^2)^{n+1}}{1-x^2}$

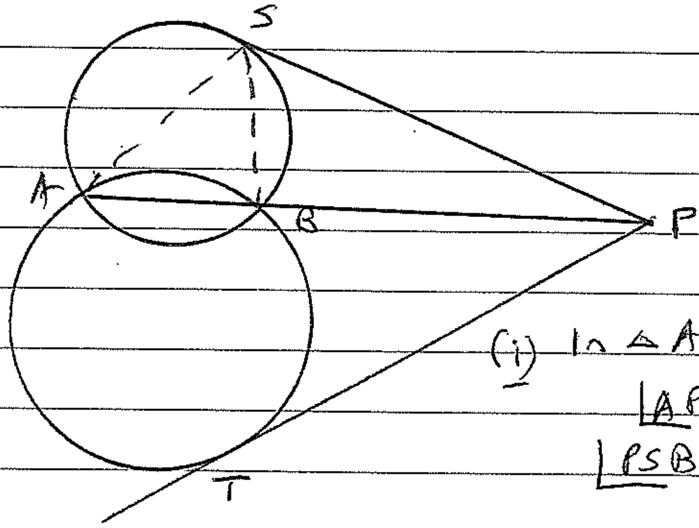
$= \frac{1-x^{2n+2}}{1-x^2}$

$S_1 \times S_2 = \frac{1-(x^{n+1})^2}{1-x^2}$

$= \frac{1-x^{2n+2}}{1-x^2} = S_3$.

(4)

(a)



(i) In $\triangle ASP$ and $\triangle SBP$

$\angle APS$ is common

$\angle PSB = \angle SAP$ (alternate angle theorem)

$\therefore \triangle ASP \parallel \triangle SBP$ (equiangular)

(ii) $\therefore$ Corresponding sides are in ratio

$\therefore \frac{SP}{BP} = \frac{AP}{SP}$

$\Rightarrow SP^2 = BP \cdot AP$.

(iii) Similarly in $\triangle ATP$ and $\triangle TBP$

$$PT^2 = AP \cdot TB.$$

OR By intercept theorem in bottom circle,

$$PT^2 = AP \cdot PB.$$

Since $PS^2 = AP \cdot TB$ (part (ii))

$$\therefore PT = PS$$

(iv) In $\triangle SPD$ and $\triangle TPD$,

$PS = PT$ (proven above)

$\angle SPD = \angle TPD$ (PD bisects $\angle SPT$)

PD is common,

$\therefore \triangle SPD \equiv \triangle TPD$ (SAS)

$\therefore \angle PSD = \angle PTD$ (corresponding angles in congruent triangles)

$= 90°$

$\therefore$ DT strikes a tangent at $90°$, So DT is part of a rad

$\therefore$ D is the centre.

(b) (i) $a = 2^N$, $r = 2^{-1}$, $ar^{n-1} = 2^N(2^{-1})^{n-1}$

$$\therefore 2^{-N} = 2^{N-n+1}$$

$$\therefore -2N = -n+1$$

$$\therefore n = 2N+1.$$

(ii)

$$S_N = \frac{2^N\left[1-(2^{-1})^{2N+1}\right]}{1/2}$$

$$= 2^{N+1}\left[1 - 2^{-1-2N}\right]$$

$$= 2^{N+1} - 2^{-N}.$$

(iii) as $N \to \infty$, $2^{N+1} \to \infty$ (while $2^{-N} \to 0$)

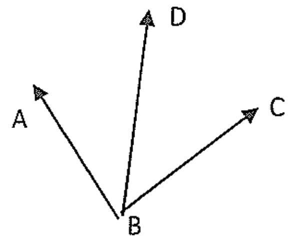
So there is no limiting sum (goes to infinity)

EXAMINERS' COMMENTS – EXTENSION 2 Paper, December 2014

QUESTION 1:

(b) Too many did not recognise that $|z-1-i|<2$ was a circle

(c) Adding vectors does not produce the same answer as adding co-ordinates. This is the VECTOR you get when you add, $\underset{BA}{\longrightarrow}$ to $\underset{BC}{\longrightarrow}$



Ie the vector BD

(d) Too many students left their answers as x = 5 and y = 1. For multiple solutions you need to separate the cases. Ie {x = 5 {y = 1 and {x = -5 {y = -1

QUESTION 2:

(a) (i) Recognise that the conjugate is between Real and Imaginary, not between the bit containing sine and cosine.

(ii) It was made easier when (i) and (ii) lead you into (iii) (NOTE this HAS been said to you before in Examiners' comments), and that (1-2cosθ + 2*i*sinθ)(1-2cosθ – 2*i*sinθ) = $(1-2cos\theta)^2+2sin\theta^2$

(b) Setting out!, setting out! Setting out! (and NO ESSAYS!)

(c) Do not leave the question in *"cis"* form

QUESTION 3:

(a) Again, if there are multiple parts (i) (ii) (iii) they are used to help each other

(b) Most students failed to recognise HOW MANY terms there were. Ie n+1 not n terms

QUESTION 4:

(a) (ii) To gain the mark you HAD to state the reason for giving the ratio of the sides (ie that they were corresponding sides in similar triangles)

(ii) The word DEDUCE is important. It doesn't mean you had to entirely redo the question. The mathematical term, "similarly" means it follows the same proof as the previous part. (NOTE: Using PT = PS because they are tangents from an external point ONLY works for the one circle, not different circles)

(iii) It is no good assuming that D is the centre of the bottom circle –n this is what you are trying to prove. Similar arguments apply to the use of cyclic quadrilaterals, and stating that the tangent hits a radius at right angles. These are both really what you are out to prove.

(b) Why didn't more people get this right? Algebra skills are poor, because this was NOT a difficult question, even though the N and the n may have confused people.

(iii) As the ratio is $\frac{1}{2}$ there should be an infinite sum. But the sum is $2^{N+1} - 2^{-N}$ will go to infinity even though the second part will go to zero)