Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 11
Mathematics Extension 2

HSC Course

Assessment 1

November, 2015

Time allowed: 70 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A set of Reference Formulae is provided. You are not to write in this booklet as it must be returned at the end of the examination.

Section 1	Multiple Choice Questions 1-5 5 Marks
Section II	Questions 6-9 50 Marks

SECTION I: MULTIPLE CHOICE

Use the multiple choice answer sheet provided, and fill in the circle corresponding with the most appropriate answer.

Allow about 7 minutes for this section.

SECTION I

1 Simplified, $\frac{1+i}{i} - \frac{i}{1-i} =$

A. $2-2i$ B $\frac{1-3i}{2}$ C. $\frac{3-3i}{2}$ D. $-2+2i$

2 The point Z, represents the complex number z.

The value of $\arg \bar{z} + \arg z$ is:

A. 0 B. $\frac{\pi}{2}$ C. π D. 2π

3. $i^{2015} =$

A. 1 B. -1 C. i D. $-i$

4. The equation $x^3 - 3x + 1 = 0$ has roots $\alpha, \beta,$ and γ.

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} =$$

A. 3 B. -3 C. 0 D. -1

5. The equation with roots $1-3i$ and $1+3i$ is:

A. $x^2 - x + 10 = 0$

B. $x^2 - x - 10 = 0$

C $x^2 - 2x + 10 = 0$

D. $x^2 - 2x - 10 = 0$

SECTION II -

Use the answer booklet provided, and start each new question on a new page.

Allow about 63 minutes for this section.

QUESTION 6: (13 Marks)

Marks

(a) For the complex number $6 - 8i$, find **4**

(i) $|z|$ (ii) $\bar{z}$ (iii) $z\bar{z}$ (iv) arg z (to the nearest minute)

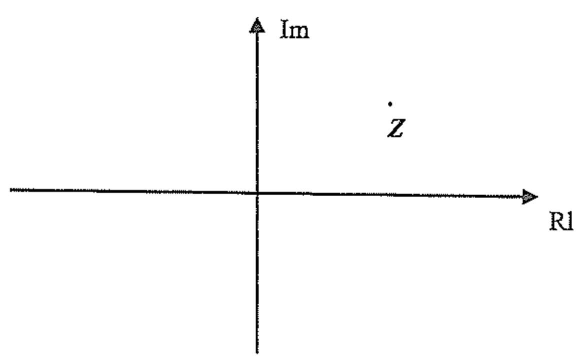
(b) Write $\frac{2-3i}{3+2i}$ in the form $a + ib$ **2**

(c) Express $\sqrt{5 + 12i}$ in the form $a + ib$ **3**

(d) (i) Express $z = 1 + i\sqrt{3}$ in mod-arg form **3**

(ii) Hence find the value of $(1 + i\sqrt{3})^9$

(e) Given that Z, representing the complex number, z, lies on the Argand Diagram, as shown below, redraw the diagram and plot the point representing $i^3 z$ **1**



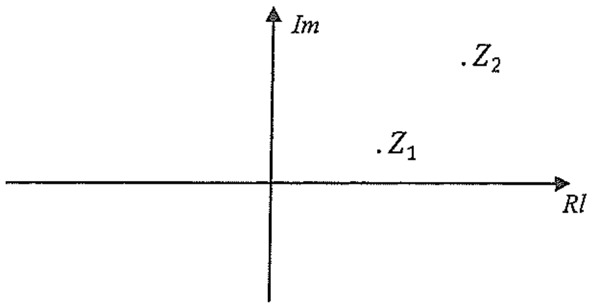
QUESTION 7: (13 Marks) (Start a new page)

Marks

(a) Given that $Im(z^2) = 2$, plot the locus of the point Z (z) on the Argand Diagram. **2**

(b) If $x = 1$ is a double root of the equation $x^4 + ax^3 + bx^2 - 5x + 1 = 0$, Find the values of a and b. **2**

(c) The points Z_1 and Z_2 representing the complex numbers z_1 and z_2 respectively, are shown on the Argand Diagram below. **2**



P (z) is a point which moves so that $\arg(z - z_1) = \arg(z - z_2)$.
Sketch the locus of the point P.

(d) Sketch the locus given by $\arg(z - 1 - i) = \frac{\pi}{4}$ **1**

(e) Suppose that the point Z representing the complex number z, lies on the unit circle, and that

$$0 \leq \arg z \leq \frac{\pi}{4}$$

(i) Sketch the locus of Z **1**

(ii) Prove that $2\arg(z + 1) = \arg z$ (give all reasons) **2**

(f) For the polynomial $4x^3 + 8x^2 + x - 3 = 0$ one root is the sum of the other 2 roots. **3**

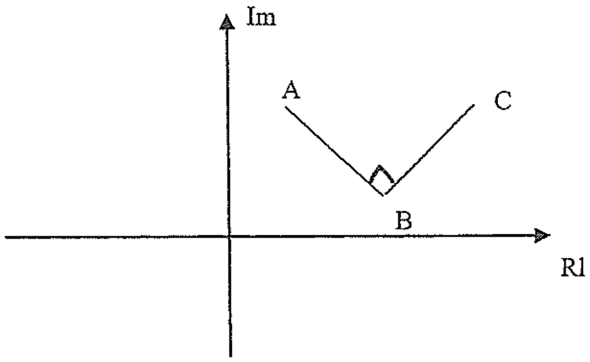
Find the values of the 3 roots.

QUESTION 8: (13 Marks) (Start a new page)

Marks

(a) In the diagram below, A, B and C represent the complex numbers z_1, z_2 and z_3 respectively

Δ ABC is isosceles and right-angled at B.



(i) Show that $(z_1 - z_2)^2 = -(z_3 - z_2)^2$ **2**

(ii) Suppose D is the point which makes ABCD a square. **1**

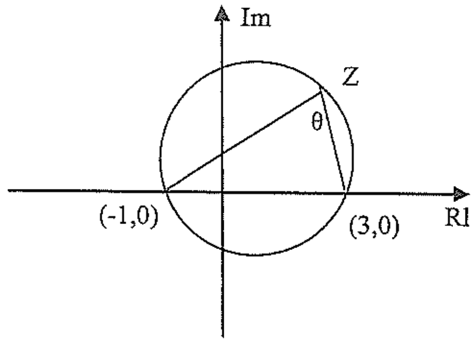
Find D in terms of z_1, z_2 and z_3

(b) Sketch the locus of the point Z representing the complex number z, if **3**

$$z\bar{z} + 2(z + \bar{z}) \leq 0$$

QUESTION 8 continues over......)

(c) Z moves on a circle which passes through the points (-1, 0) and (3, 0) as shown below. **3**



It is given that $arg\left[\frac{z-3}{z+1}\right] = \frac{\pi}{3}$.

Find the value of θ and the y-value of the centre of the circle.

(d) P represents the complex number z, where $|z-2| = 1$

If O is the origin, find:

(i) the minimum distance OP? **1**

(ii) the maximum distance OP? **1**

(iii) the largest value of arg z ? **2**

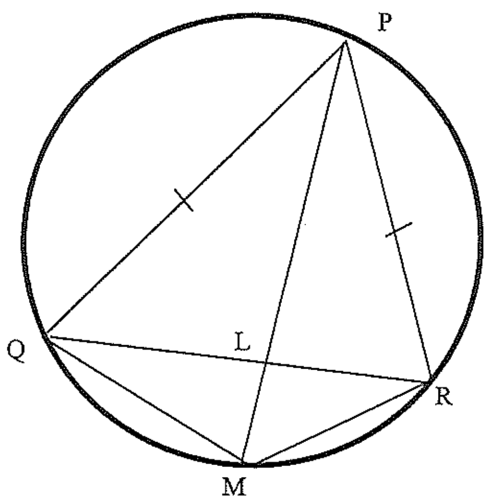
QUESTION 9: 11 Marks (Start a New Page)

(a) (i) For $k \geq 1$, prove that $\frac{1}{k} - \frac{1}{k+1} > \frac{1}{(k+1)^2}$ **2**

(ii) Prove, by the process of mathematical induction, that for $n \geq 1$, **4**

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots \ldots\ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

(b)



PQR is an isosceles triangle with PQ = PR.

L is the point of intersection of the diagonals of the cyclic quadrilateral PRMQ.

(i) Prove that ΔQLM is similar to ΔPLR **2**

(ii) Show that $QM \times LR = LM \times PR$ **1**

(iii) It can be further proved that ΔQLP is similar to ΔMLR which leads to the statement that $MR \times QL = QP \times LM$ **2**
(You do not have to prove this. It may be assumed for the next section.)

Prove that:

$$\frac{1}{QM} + \frac{1}{MR} = \frac{QR}{LM \times PR}$$

SOLUTIONS

QUESTIONS 1 to 5

1/ C 2/ A 3/ D 4/ A 5/ C

QUESTION 6:

(a) (i) 10 (ii) $6+8i$ (iii) 100

(iv) $\tan\theta = -8/6$

$\theta = -53°8'$ (OR $306°52'$)

(b) $\frac{2-3i}{3+2i} \times \frac{3-2i}{3-2i} = -i$

(c) $x^2 - y^2 + 2ixy = 5+12i$

$x^2 - y^2 = 5$ and $xy = 6$

$y = 6/x$.

$\therefore x^2 - (6/x)^2 = 5$

$x^4 - 5x^2 - 36 = 0$

$(x^2-9)(x^2+4) = 0$

$x = \pm 3$ or, $x = \pm 2i$

If $x = 3$, $y = 2$

$\sqrt{5+12i} = \pm(3+2i)$

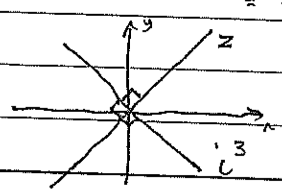
(d) (i) $r = 2$, $\tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$

$\therefore z = 2\,\text{cis}\,\pi/3$

(ii) $z^9 = 2^9\,\text{cis}\,3\pi$

$= -512$ or, $512\,\text{cis}\,\pi$

(e)



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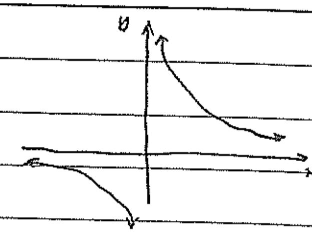
QUESTION 7 (a) $\text{Im}(z^2) = 2$

$\therefore \text{Im}(x^2 - y^2 + 2ixy) = 2$

$xy = 1$



(b) $P(x) = x^4 + ax^3 + bx^2 - 5x + 1$

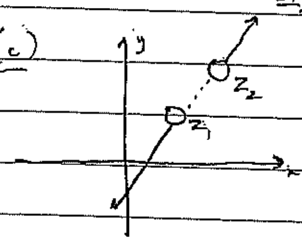
$P'(x) = 4x^3 + 3ax^2 + 2bx - 5$

Now $P(1) = 0 \Rightarrow a + b = 3$

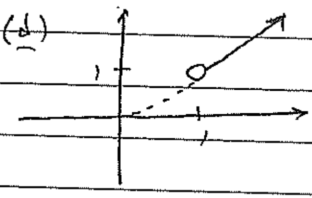
$P'(1) = 0 \Rightarrow 3a + 2b = 1$

$\Rightarrow a = -5 \quad b = 8$

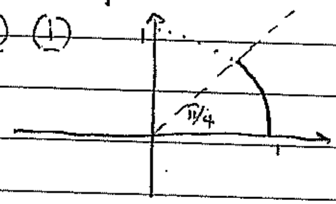
(c)



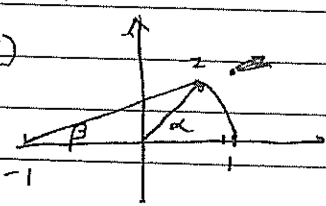
(d)



(e) (i)



(ii)



Let $\arg z = \alpha$ and $\arg(z+1) = \beta$

The triangle formed is isosceles (because z is on unit circle)

$\therefore \alpha = 2\beta$ (exterior angle theorem)

(f) $\gamma = \alpha + \beta$

Sum of Roots $= 2(\alpha+\beta) = -2$

$\therefore \alpha + \beta = -1$ (1)

Product: $\alpha\beta(\alpha+\beta) = 3/4$

$\therefore \alpha\beta = -3/4$ (2)

Solving (1) and (2) gives

$\alpha = 1/2$ or, $\alpha = -3/2$

$\therefore$ Roots are $1/2, -3/2, -1$

QUESTION 8:

(a) (i) $z_1 - z_2 = i(z_3 - z_2)$ (ii) $z_1 + z_3 - z_2$

$(z_1 - z_2)^2 = i^2(z_3 - z_2)^2$

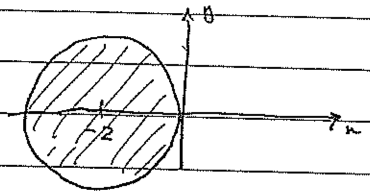
$= -(z_3 - z_2)^2$

(b) Let $z = x + iy$

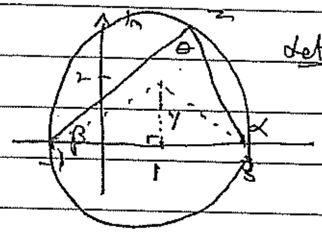
$\bar{z} = x - iy$ $z\bar{z} + 2(z + \bar{z}) \leq 0$

$\Rightarrow x^2 + y^2 + 4x \leq 0$

$\therefore (x+2)^2 + y^2 \leq 1$



(c)



Let $\arg(z - 3) = \alpha$

and $\arg(z + 1) = \beta$

$\therefore \arg(z-3) - \arg(z+1) = \theta$

$\therefore \arg\left(\frac{z-3}{z+1}\right) = \theta$

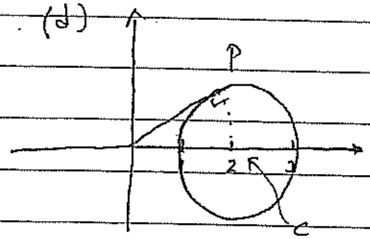
$\therefore \theta = \pi/3$

(ii) $y/2 = \frac{1}{\tan \pi/3}$

$\therefore y = 2/\sqrt{3}$

← because the y-line bisects the angle ~~AND that angle is 2θ~~
(angle at the centre is twice that on the circumference)

(d)



(i) MIN $OP = 1$

(ii) MAX $OP = 3$

(iii) Largest value of $\arg z$ is when OP is a tangent. This means that $OP \perp PC$. $\therefore PC = 1$ and OC is 2

If $\arg z = \theta$

$\sin\theta = 1/2 \Rightarrow \theta = \pi/6$.

QUESTION 9:

(a) (i) $\frac{1}{k} - \frac{1}{k+1} = \frac{k+1-k}{k(k+1)}$

$= \frac{1}{k^2+k}$

$> \frac{1}{(k+1)^2}$ because $k^2+2k+1 > k^2+k$

(ii) For $n = 1$ LHS $= 1$, RHS $= 1$

~~$\therefore$ LHS = RHS $\therefore$ trivial (proves =)~~

For $n = 2$ LHS $= 5/4$ RHS $= 2 - 1/2$

$= 3/2$

$\therefore$ LHS < RHS

$\therefore$ ~~the formula is true for $n = 1, 2$~~

~~Assume the formula is true for $n = k$~~

$\therefore 1 + 1/4 + 1/9 + \ldots + 1/k^2 \leq 2 - 1/k$

For $n = k+1$

$1 + 1/4 + 1/9 + \ldots + 1/k^2 + \frac{1}{(k+1)^2} \leq 2 - 1/k + \frac{1}{(k+1)^2}$

$= 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right)$

$< 2 - \frac{1}{k+1}$

from the result in part (i)

Which is of the same form as for $n = k$

$\therefore$ If it is true for $n = k$, it is true for $n = k+1$

But the formula is true for $n = 2$.

$\therefore$ " " " " $n = 3$

and so on.

$\therefore$ true for all n.

QUESTION 9 (b)

(i) In $\triangle QLM$ and $\triangle PLR$

$\angle MQL = \angle LPR$ (angles at the circumference ~~standing on the arc MR~~)

$\angle QLM = \angle PLR$ (vertically opposite angles)

$\therefore \triangle QLM \sim \triangle PLR$ (equiangular)

(ii) $\dfrac{QM}{LM} = \dfrac{PR}{LR}$ (corresponding sides in similar triangles are in ratio)

$\therefore QM \times LR = LM \times PR \quad (1)$

(iii) Similarly $MR \times QL = QP \times LM \quad (2)$

From (1) $QM = \dfrac{LM \times PR}{LR}$

From (2) $MR = \dfrac{QP \times LM}{QL}$

$\therefore \dfrac{1}{QM} + \dfrac{1}{MR} = \dfrac{LR}{LM.PR} + \dfrac{QL}{LM.PQ}$

and since $PQ = PR$

$\therefore \dfrac{1}{QM} + \dfrac{1}{MR} = \dfrac{LR + QL}{LM.PR}$

$= \dfrac{QR}{LM.PR}$

EXTENSION 2 MARKERS' COMMENTS – December 2015

QUESTION 6:

Well answered. See your teacher for handy hints on the method of finding square roots.

QUESTION 7:

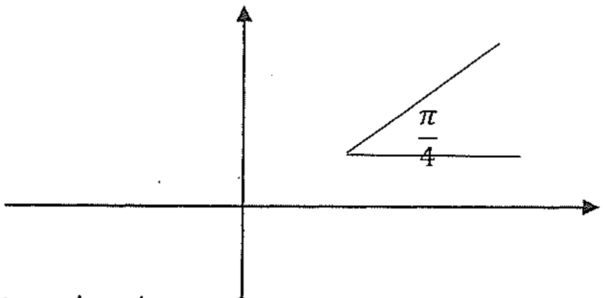
Locus needs lots of work.

(a) $Im(z^2) = 2$ means you have to find z^2 first

(b) When the words "double root" appear, you can assume you have to use $P(a)$, $P'(a)$

(c) Learn the formats of locus. Use of an open circle at the ends of a line, when z lies on a point. You can't have the argument of a point.

(d) Understood the question, but the habit of showing the angle of $\frac{\pi}{4}$ like this is that it indicates that the horizontal line is part of the locus:



(e)(i) In this question z moved on the arc of a unit circle. It did not include a shaded region.

(e) (ii) It is often easier to draw a picture to cover your explanation. In this part a lot of people "assumed" that arg(z) = $\frac{\pi}{4}$, which was not the case. Z was a variable

QUESTION 8:

(a) Multiplication by i causes a rotation through 90 anticlockwise. A lot of students wrote BC = i X BA, which is not true.

(a)(ii) Very poorly done overall

(b) Some very simple algebraic errors here.

(c) Most students had trouble finding the centre.

(d) Most students used tan rather than sin to find arg z.

QUESTION 9:

This was a very difficult question, made to "sort out" the pack.

(a) (i)This is just an algebraic "show". Don't use Mathematical Induction unless specifically asked to by the question.

(a)(ii) A tough Induction question. Even the Mathematics staff differ in opinion as to whether:

You have to show that the inequality is true for n=2, since n=1 only shows the equality

OR You only have to show it is true for n=1, because the algebra in step 2 takes care of the inequality.

(We are seeking clarification from Universities)

YOU WERE NOT PENALISED IF YOU ONLY SHOWED THAT IT WAS TRUE FOR n= 1.

Marks were not awarded for the conclusion if the rest of the algebraic part of the proof was missing, badly done.

The whole point of having part (i) was that it was to be used in part (ii), when the algebra became difficult.

(b) (iii) Very difficult. Using line lengths as algebra was too hard for most people. See your solutions.