**GIRRAWEEN HIGH SCHOOL**

# MATHEMATICS

**YEAR 12 HSC** **Task 2, 2012**

**Time Allowed: 90 minutes** **Name:**_____

Instructions: Examiner: C. McMillan

- Attempt all questions
- Circle the best response for the questions in Part A
- Start each question in Part B on a new page
- All necessary working must be shown
- Marks may be deducted for careless or badly arranged work

## PART A (5 marks)

**For questions 1-5 circle the best response from the following:**

**Question 1:** The probability that a traffic light will turn green as a vehicle approaches it is estimated to be $\frac{5}{12}$. A taxi goes through 192 intersections where there are traffic lights. How many of these would be expected to turn green as the taxi approaches:

A) 80 B) 112 C) 110 D) 192

**Question 2:** All $x$ values for which the curve $f(x)=x^2-4x+1$ is increasing are:

A) $x\leq 2$ B) $x\geq 2$ C) $x<2$ D) $x>2$

**Question 3:** Given $h=5t^3-2t^2+t+5$, the value of $\frac{d^2h}{dt^2}$ when $t=1$ is:

A) 9 B) 12 C) 26 D) 27

**Question 4:** The primitive function of $(2x-1)^2$ is:

A) $\frac{(2x-1)^3}{6}+C$ B) $\frac{(2x-1)^3}{3}+C$ C) $\frac{(2x-1)^3}{2}+C$ D) $(2x-1)^3+C$

**Question 5:** The value of $\int_{-1}^{0} x^3 dx$ is:

A) $\frac{1}{4}$ B) $-\frac{1}{4}$ C) 1 D) -1

## PART B

**Question 1 (11 marks)**

(a) A book has 124 pages. If the book is opened at any page at random, find the probability of the page number being:

i) either 80 or 90 (1)
ii) a multiple of 10 (2)
iii) an odd number (1)
iv) less than 100 (1)

(b) In a group of 75 students, altogether 54 do History and 31 do Geography. If I select one student at random, find the probability that the student will do:

i) only Geography (2)
ii) both History and Geography (2)

(c) In Yahtzee, 5 dice are rolled. Find the probability of rolling five 6's (2)

**Question 2 (11 marks)**

(a) Find any stationary points on the curve $f(x)=2x^3-15x^2+24x-7$ and determine their nature. (5)

(b) Find all values of $x$ for which the curve is concave up given that $y=x^3-x^2+x+9$. (3)

(c) If $y=x^2$ show that $2y\left(\frac{d^2y}{dx^2}\right)=\left(\frac{dy}{dx}\right)^2$. (3)

**Question 3 (9 marks)**

(a) Find the second derivative of $y=\sqrt{3x-1}$. (3)

(b) Sketch $f(x)=2x^3+3x^2-36x+5$ for $-3\le x\le 3$, showing any stationary points and points of inflexion. Find the maximum and minimum values of the function in $-3\le x\le 3$. (6)

**Question 4 (11 marks)**

(a) The council wanted to make a rectangular swimming area at the beach using a straight cliff on one side and a length of 300m of shark proof netting for the other three sides. What are the dimensions of the rectangle that encloses the greatest area. (5)

(b) A poster consists of a photograph bordered by a 5cm margin. The area of the poster is to be $400cm^2$.

(i) Show that the area of the photograph is given by

$$A = 500 - 10x - \frac{4000}{x}$$ (3)

(ii) Find the maximum area possible for the photograph. (3)

**Question 5 (22 marks)**

(a) Find each indefinite integral:

(i) $\int (y-3)dy$ (1)

(ii) $\int \frac{dx}{x^3}$ (2)

(iii) $\int \sqrt{x}\left(1+\frac{1}{\sqrt{x}}\right)dx$ (3)

(iv) $\int (4+3x)^4 dx$ (1)

(v) $\int \sqrt{(5x+2)^5}\,dx$ (3)

(b) Evaluate:

(i) $\int_0^2 \left(\frac{4x^3+x^2+5x}{x}\right)dx$ (4)

(ii) $\int_1^4 \sqrt{x}dx$ (3)

(iii) $\int_4^5 (5-x)^6 dx$ (2)

(iv) $\int_0^1 \frac{dx}{(3x-2)^4}$ (3)

**Question 6 (12 marks)**

(a) If $y = ax^3 - 12x^2 + 3x - 5$ has a point of inflexion at $x = 2$, evaluate $a$. (3)

(b) Given that the gradient of the tangent to a curve is given by $\frac{dy}{dx} = 2 - 6x$ and the curve passes through $(-2, 3)$, find the equation of the curve. (3)

(c) A function has a tangent parallel to the line $4x - y - 2 = 0$ at the point $(0, -2)$ and $f''(x) = 12x^2 - 6x + 4$. Find the equation of the function. (6)

**END OF PAPER.**

## PART A

1) A
2) D
3) C
4) A
5) B

## PART B

### Question 1

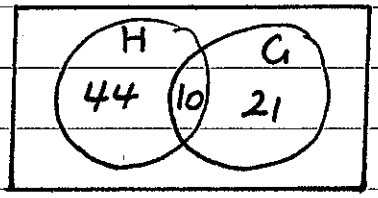
ai) $P(\text{either 80 or 90}) = \frac{1}{62}$

ii) $P(\times 10) = \frac{12}{124} = \frac{3}{31}$

iii) $P(\text{odd number}) = \frac{1}{2}$

iv) $P(<100) = \frac{99}{124}$

b)




i) $P(\text{only Geography}) = \frac{21}{75} = \frac{7}{25}$

ii) $P(\text{both History and Geography}) = \frac{2}{15}$

c) $P(\text{five 6's}) = \left(\frac{1}{6}\right)^5 = \frac{1}{7776}$

### Question 2

a) $f'(x) = 6x^2 - 30x + 24$

$6(x^2 - 5x + 4) = 0$

$\therefore (x-4)(x-1) = 0$

$\therefore x = 4, 1$

Stationary pts.

When $x = 4$, $y = 2(4)^3 - 15(4)^2 + 24(4$
$-7$

$= -23$

When $x = 1$, $y = 2(1)^3 - 15(1)^2 + 24(1)-$
$= 4$

$\therefore$ stat. pts at $(4, -23)$, $(1, 4)$

$f''(x) = 12x - 30$

When $x = 4$, $f''(x) > 0$ $\therefore$ min. turning pt

$x = 1$, $f''(x) < 0$ $\therefore$ max. turning pt

$\therefore (4, -23)$ is a minimum turning pt and $(1, 4)$ is a maximum turning pt.

b) $y' = 3x^2 - 2x + 1$

$y'' = 6x - 2$

$y'' > 0$

$6x - 2 > 0$

$6x > 2$ $\therefore x > \frac{1}{3}$

## Question 2 cont.

c) $y = x^2$

$\frac{dy}{dx} = 2x$

$\frac{d^2y}{dx^2} = 2$

$2y\left(\frac{d^2y}{dx^2}\right) = \left(\frac{dy}{dx}\right)^2$

$LHS = 2y\left(\frac{d^2y}{dx^2}\right)$

$= 2(x^2)(2)$

$= 4x^2$

$RHS = \left(\frac{dy}{dx}\right)^2$

$= (2x)^2$

$= 4x^2$

$= LHS$

$\therefore 2y\left(\frac{d^2y}{dx^2}\right) = \left(\frac{dy}{dx}\right)^2$

## Question 3

a) $y = \sqrt{3x-1}$

$= (3x-1)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(3x-1)^{-\frac{1}{2}} \times 3$

$= \frac{3}{2}(3x-1)^{-\frac{1}{2}}$

$= \frac{3}{2\sqrt{3x-1}}$

$\frac{d^2y}{dx^2} = -\frac{3}{4}(3x-1)^{-\frac{3}{2}} \times 3$

$= -\frac{9}{4}(3x-1)^{-\frac{3}{2}}$

$= -\frac{9}{4\sqrt{(3x-1)^3}}$

## Question 3

a) $y' = 3x^2 - 12x + 5$

$y'' = 6x - 12$

pt of inflexion when $y'' = 0$

$6x - 12 = 0$

$6x = 12$

$x = 2$ possible pt of inflexion

Test:

| $x$ | 1.9 | 2 | 2.1 |
|---|---|---|---|
| $y''$ | -0.6 | 0 | 0.6 |

When $x = 2$

$y = 2^3 - 6(2)^2 + 5(2) + 9$

$= 8 - 24 + 10 + 9$

$= 3$

∴ A point of inflexion is at $(2, 3)$.

b) Stationary pts.

$f'(x) = 6x^2 + 6x - 36$

$6(x^2 + x - 6) = 0$

$\therefore (x+3)(x-2) = 0$

$\therefore x = -3, 2$

When $x = -3$

$f(x) = 2(-3)^3 + 3(-3)^2 - 36(-3) + 5$

$= 86$

When $x = 2$

$f(x) = 2(2)^3 + 3(2)^2 - 36(2) + 5$

$= -39$

When $x = 3$

$f(x) = 2(3)^3 + 3(3)^2 - 36(3) + 5 = -22$

$f''(x) = 12x + 6$

When $x = -3$, $f''(x) < 0$

maximum

When $x = 2$, $f''(x) > 0$ ∴ minimum

∴ $(-3, 86)$ maximum turning pt and $(2, -39)$ minimum turning pt

$f''(x) = 0$

$12x + 6 = 0$

$\therefore x = -\frac{1}{2}$ possible pt of inflexion

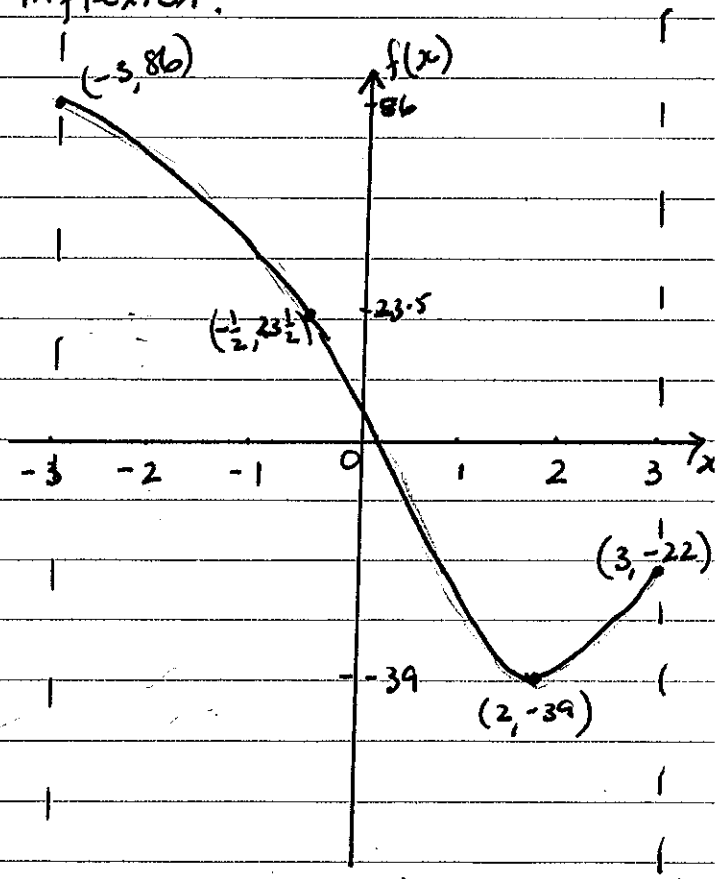
Test:

| $x$ | -0.51 | -0.5 | -0.49 |
|---|---|---|---|
| $f''(x)$ | -0.12 | 0 | 0.12 |

When $x = -0.5$

$f(x) = 23.5$

∴ $(-0.5, 23.5)$ is a point of inflexion.

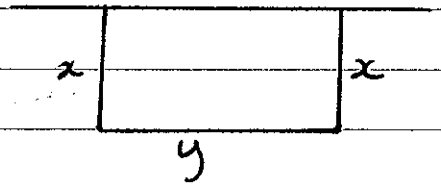



∴ minimum value = -39

maximum value = 86

# Question 4.

a)




Let:
Width be $x$
length be $y$.

Perimeter: $300 = 2x + y$

$\therefore y = 300 - 2x$

Area $= x(300 - 2x)$

$= 300x - 2x^2$

$\frac{dA}{dx} = 300 - 4x$

When $\frac{dA}{dx} = 0$

$300 - 4x = 0$

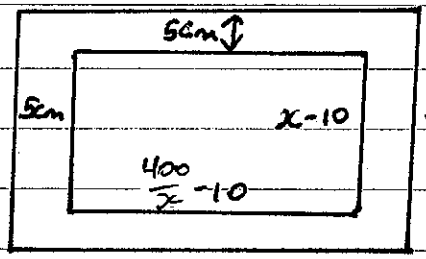
$4x = 300$

$x = 75$

$\frac{d^2A}{dt^2} = -4$ always max.

$\therefore$ The width is 75m and the length is 150m.

b)




Poster

$A = 400cm^2$

$\therefore xy = 400$

$\therefore y = \frac{400}{x}$

Photo Area $= \left(\frac{400}{x} - 10\right)(x - 10)$

$= 400 - \frac{4000}{x} - 10x + 100$

$= 500 - \frac{4000}{x} - 10x$

$\therefore A = 500 - 10x - \frac{4000}{x}$

ii) $\frac{dA}{dx} = -10 + \frac{4000}{x^2}$

$-10 + \frac{4000}{x^2} = 0.$

$\frac{4000}{x^2} = 10$

$10x^2 = 4000$

$x^2 = 400$

$\therefore x = 20$

$\frac{d^2A}{dx^2} = -\frac{4000}{x^3}$

When $x = 20$, $\frac{d^2A}{dx^2} < 0$

$\therefore$ maximum.

Area Photo $= 500 - 200 - 200$

$= 100cm^2$

## Question 5

ai) $\int (y-3)\,dy$

$= \frac{y^2}{2} - 3y + c$

ii) $\int \frac{dx}{x^3}$

$= \int x^{-3}\,dx$

$= -\frac{x^{-2}}{2}$

$= -\frac{1}{2x^2} + c$

iii) $\int \sqrt{x}\left(1 + \frac{1}{\sqrt{x}}\right) dx$

$= \int \sqrt{x} + 1\,dx$

$= \int x^{\frac{1}{2}} + 1\,dx$

$= \frac{2}{3}x^{\frac{3}{2}} + x$

$= \frac{2}{3}\sqrt{x^3} + x$

$= \frac{2x\sqrt{x}}{3} + x + c$

iv) $\int (4+3x)^4\,dx$

$= \frac{(4+3x)^5}{15} + c$

v) $\int \sqrt{(5x+2)^5}\,dx$

$= \int (5x+2)^{\frac{5}{2}}\,dx$

$= \frac{(5x+2)^{\frac{7}{2}}}{5 \times \frac{7}{2}}$

$= \frac{2\sqrt{(5x+2)^7}}{35} + c$

bi) $\int_0^2 \left(\frac{4x^3 + x^2 + 5x}{x}\right) dx$

$= \int_0^2 4x^2 + x + 5\,dx.$

$= \left[\frac{4x^3}{3} + \frac{x^2}{2} + 5x\right]_0^2$

$= \left(\frac{32}{3} + 2 + 10\right) - 0$

$= \frac{68}{3}$

ii) $\int_1^4 \sqrt{x}\,dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$

$= \left(\frac{2}{3}(4)^{\frac{3}{2}}\right) - \left(\frac{2}{3}(1)^{\frac{3}{2}}\right)$

$= \frac{14}{3}$

## Question 5 cont.

iii) $\int_4^5 (5-x)^6 \, dx$

$= \left[ -\frac{(5-x)^7}{7} \right]_4^5$

$= +\frac{1}{7}$

iv) $\int_0^1 \frac{dx}{(3x-2)^4} \, dx$

$= \int_0^1 (3x-2)^{-4} \, dx$

$= \left[ -\frac{(3x-2)^{-3}}{9} \right]_0^1$

$= \left[ -\frac{1}{9(3x-2)^3} \right]_0^1$

$= \left(-\frac{1}{9}\right) - \left(\frac{1}{72}\right)$

$= -\frac{9}{72} = -\frac{1}{8}$

## Question 6

a) $y' = 3ax^2 - 24x + 3$

$y'' = 6ax - 24$

$y'' = 0$ at $x = 2$

$0 = 12a - 24$

$12a = 24$

$\therefore a = 2$

b) $\frac{dy}{dx} = 2 - 6x$

$y = 2x - 3x^2 + c$

passes through $(-2, 3)$

$3 = 2(-2) - 3(-2)^2 + c$

$3 = -4 - 12 + c$

$\therefore c = 19$

$\therefore y = 19 + 2x - 3x^2$

c) $4x - y - 2 = 0$

$\therefore y = 4x - 2$

$m_{tangent} = 4$ at $(0, -2)$

$f''(x) = 12x^2 - 6x + 4$

$f'(x) = 4x^3 - 3x^2 + 4x + c$

$f'(x) = 4$ at $(0, -2)$

$4 = \frac{12(0)^3}{3} - 3(0)^2 + 4(0) + c$

$\therefore c = 4$

$\therefore f'(x) = 4x^3 - 3x^2 + 4x + 4$

$f(x) = x^4 - x^3 + 2x^2 + 4x + c$

when $x = 0$ $f(x) = -2$.

$-2 = 0^4 - 0^3 + 2(0)^2 + 4(0) + c$

$\therefore c = -2$.

$\therefore f(x) = x^4 - x^3 + 2x^2 + 4x - 2$.