GIRRAWEEN HIGH SCHOOL
MATHEMATICS

YEAR 12 HSC **Task 2 2013**

Time Allowed: 90 minutes

INSTRUCTIONS TO STUDENTS

- Attempt **ALL 6** questions.
- Circle the best response for the questions in **Part A**
- Start each question in **Part B** on a new page.
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.

PART A (5 marks)

For questions 1-5 circle the best response from the following:

Question1. The chance of a fisherman catching a legal length fish is 4 in 5. If three fish are caught at random, what is the probability that exactly one is of legal length?

(A) $\frac{4}{125}$ (B) $\frac{12}{125}$ (C) $\frac{16}{125}$ (D) $\frac{48}{125}$

Question 2. For which values of x is the curve $f(x)=2x^3+x^2$ concave down?

(A) $x<-\frac{1}{6}$ (B) $x>-\frac{1}{6}$ (C) $x<-6$ (D) $x>6$

Question 3. The graph $y=f(x)$ passes through the point $(1,4)$ and $f'(x)=3x^2-2$. Which of the following expressions is $f(x)$.

(A) x^3-2x (B) $2x-1$ (C) x^3-2x+3 (D x^3-2x+5

Question 4. What is the value of $\int_{-1}^{2} x^2+1dx$?

(A) 4 (B) 5 (C) 6 (D 7

Question 5. The primitive function of $\frac{1}{(2x+3)^2}$ is:

(A). $\frac{-1}{2(2x+3)}+C$ (B) $\frac{-1}{(2x+3)}+C$ (C) $\frac{1}{(x+3)}+C$ (D) $\frac{-2}{(x+3)}+C$

QUESTION 1 (11 marks)

(a) A box has 4 Geography books and 3 Mathematics books in it.
Two books are selected at random from the box.

i. Draw a tree diagram to show all the possible outcomes. **2**

ii. Find the probability that:

α. the two books are Mathematics books. **2**
β. at least one of the books is a Geography book **1**

(b) In a group of 80 students, 65 study maths, 45 study science and 5 do not study either maths or science. What is the probability that the student:

(i) studies maths and science. **1**

(ii) studies maths or science. **1**

(iii) studies only one of the two subjects. **2**

(iv) does not study science. **2**

QUESTION 2 (12 marks)

Consider the curve given by the equation $y = 9x(x-2)^2$.

i) Find the co-ordinates of the stationary points and determine their nature. **5**

ii) Find the co-ordinates of any points of inflexion. **2**

iii) Sketch the curve in the domain $-1 \le x \le 3$. **3**

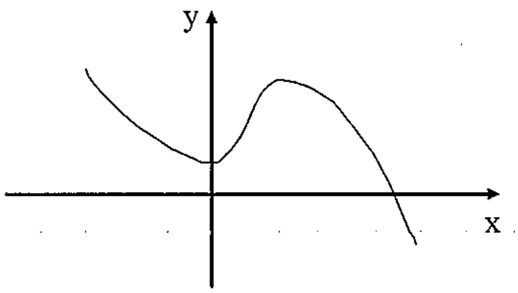
iv) What is the maximum value of $9x(x-2)^2$ in the

domain $-1 \le x \le 3$ **2**

Question 3. (9 marks)

(a) Find the range of values of x for which the curve $y = 2x^3 - 3x^2 - 12x + 8$ is concave up **2**

(b) The diagram shows the graph of a certain function f(x).



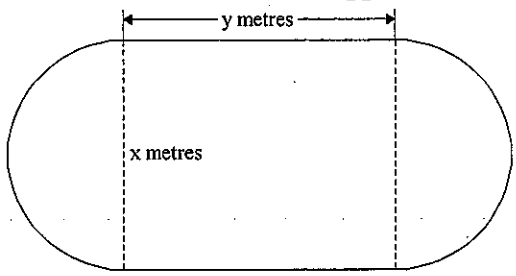
Copy this graph into your writing booklet. On the same set of axes, draw a sketch of the derivative f ′(x) of the function. **4**

(c) Prove that $f(x) = 2x^3 - 3x^2 + 5x + 1$ has no stationary points. **2**

QUESTION 4 (11marks)

(a) Find the value of k such that $\int_1^k \frac{4}{x^2}dx = 3$ **3**

(b) A railway enthusiast designs a miniature railway of length 1000 metres. The route consists of two semicircles at opposite ends of a rectangle.



i.) If the rectangle has a length of y metres and its width is x metres, show that:

$$y = 500 - \frac{\pi x}{2}.$$ **2**

ii.) Show that the area, A, enclosed by the railway track is given by $$A = \frac{2000x - \pi x^2}{4}.$$ **3**

iii.) Find the maximum area, to the nearest hectare, enclosed by the railway track. **3**

Question 5.(23 marks)

(a) Find each indefinite integral:

(i) $\int \frac{dx}{x^5}$ **1**

(ii) $\int x^2(5-3x)dx$ **2**

(iii) $\int \frac{dx}{\sqrt{9-2x}}$ **3**

(iv) $\int (3x+1)^4 dx$ **2**

(v) $\int \frac{2x^3 - x^4}{4x} dx$ **3**

(b) Evaluate:

(i) $\int_1^9 (1+\sqrt{x})\ dx$ **3**

(ii) $\int_2^3 (2x-5)^3 dx$ **3**

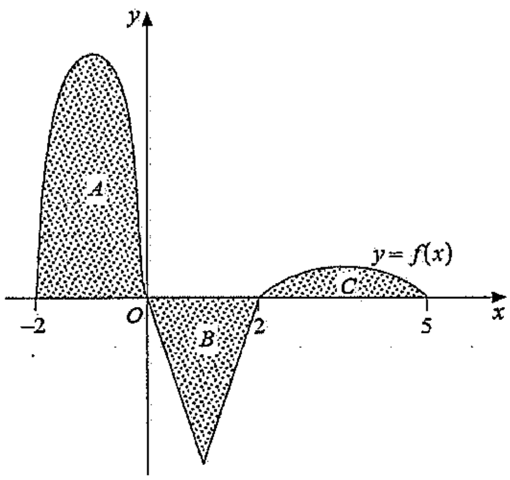
(iii) $\int_0^1 \frac{x^3 - 2x^2 + 3x}{x} dx$ **3**

(iv) $\int_1^2 \frac{dx}{(3x-1)^2}$ **3**

Question 6.(11 marks)

(a) The curve $y = f(x)$ has gradient function $\frac{dy}{dx} = 3x^2 - 2x + 1$. The curve passes through the point Q(2, 3). Find its equation. **3**

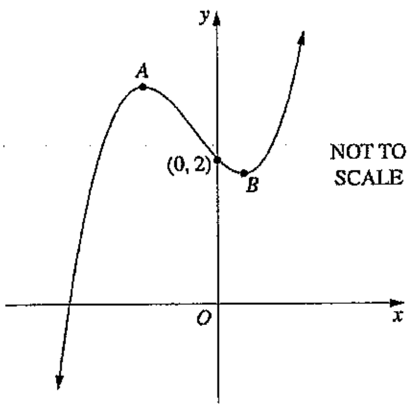
(b)



The graph of the function f is shown in the diagram. The shaded areas are bounded by $y = f(x)$ and the x axis. The shaded area A is 8 square units, the shaded area B is 3 square units and the shaded area C is 1 square unit.

Evaluate $\int_{-2}^{5} f(x)dx.$ **3**

(b) The graph of $y = x^3 + x^2 - x + 2$ is sketched below. The points A and B are the turning points.



(i) Find the coordinates of A and B. **3**

(ii) For what values of x is the curve concave up? Give reasons for your answer. **2**

End of Test

Year 12 mathematics Task 2 2013 Solutions

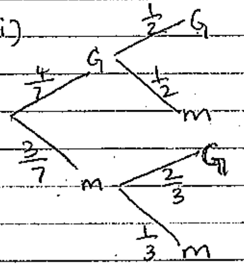
Part A

1) B
2) A
3) D
4) C
5) A

Part B

Question 1

a) (i)

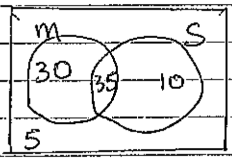


(2)

(ii) α) $\frac{3}{7} \times \frac{1}{3} = \frac{1}{7}$ (2)

(iii) β) $1 - \frac{1}{7} = \frac{6}{7}$ (1)

b)



(i) $\frac{35}{80} = \frac{7}{16}$ (1)

(ii) $\frac{75}{80} = \frac{15}{16}$ (1)

(iii) $\frac{40}{80} = \frac{1}{2}$ (2)

(iv) $\frac{35}{80} = \frac{7}{16}$ (2)

Question 2

(i) $y = 9x(x-2)^2$

$\frac{dy}{dx} = 9\left[2x(x-2) + (x-2)^2\right]$

$= 9(x-2)\left[2x + x - 2\right]$

$= 9(x-2)(3x-2)$ (2)

For stationary points, $\frac{dy}{dx} = 0 \Rightarrow 9(x-2)(3x-2) = 0$

$\therefore x = 2, \frac{2}{3}$ (1)

When $x = 2$, $y = 9 \cdot 2 \cdot 0 = 0$

when $x = \frac{2}{3}$, $y = 9 \cdot \frac{2}{3} \cdot \frac{16}{9}$

$= 10\frac{2}{3}$

$\therefore$ stationary points are $(2, 0)$ and $(\frac{2}{3}, 10\frac{2}{3})$ (2)

(ii) $\frac{dy}{dx} = 9(x^2 - 8x + 4)$

For possible points of inflexion, $\frac{d^2y}{dx^2} = 0$

$\frac{d^2y}{dx^2} = 6x - 8 = 0 \Rightarrow x = \frac{4}{3}$

Concavity test.

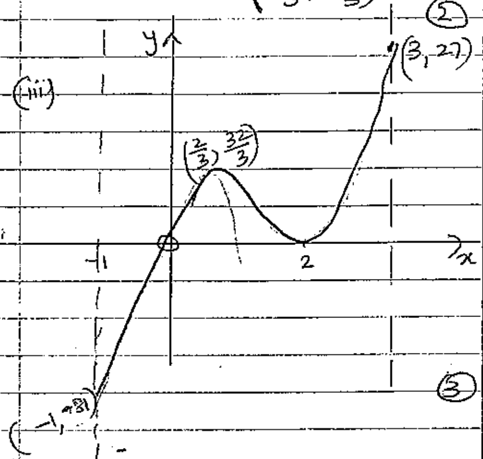
x	1	$\frac{4}{3}$	2
y''	-2	0	4

$\therefore$ Concavity changes when $x = \frac{4}{3}$

when $x = \frac{4}{3}$, $y = \frac{16}{3}$

$\therefore$ point of inflexion at $(\frac{4}{3}, \frac{16}{3})$ (2)

(iii)



(3)

(iv) When $x = 3$, $y = 27$

$\therefore$ maximum value $= 27$ (2)

Question 3

a) $y = 2x^3 - 3x^2 - 12x + 8$

$y' = 6x^2 - 6x - 12$

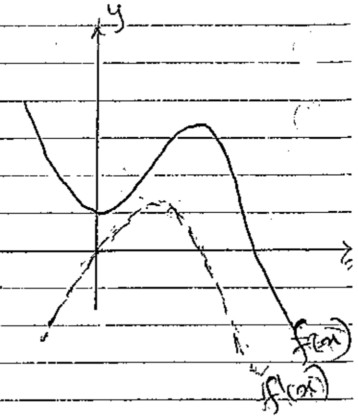
$y'' = 12x - 6$

Concave up $\Rightarrow y'' > 0$

$\therefore 12x - 6 > 0$

$\therefore x > \frac{1}{2}$ (2)

b)



(4)

(c) $f(x) = 2x^3 - 3x^2 + 5x + 1$

$f'(x) = 6x^2 - 6x + 5$

For stationary points, $f'(x) = 0$

$\therefore 6x^2 - 6x + 5 = 0$ has no solutions as $\Delta < 0$

$\therefore$ has no stationary point (2)

Question 4

(a) $\int_1^k \frac{4}{x^2}\,dx = 3$

$\int_1^k \frac{4}{x^2}\,dx = \int_1^k 4x^{-2}\,dx$

$= \left[\frac{4\cdot x^{-1}}{-1}\right]_1^k = -4\left[\frac{1}{x}\right]_1^k$

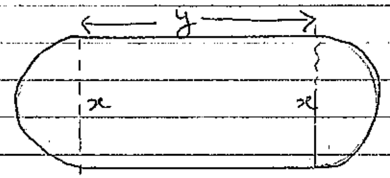
$= -4\left[\frac{1}{k} - 1\right] = 3$

$\therefore \frac{1}{k} - 1 = -\frac{3}{4}$

$\frac{1}{k} = \frac{1}{4} \Rightarrow k = 4$

(3)

(b)



(i) $\pi x + 2y = 1000$

$\therefore y = \frac{1000 - \pi x}{2}$

$= 500 - \frac{\pi x}{2}$

(2)

(ii) Area $A = xy + \frac{\pi \cdot x^2}{4}$

$= x\cdot\left(500 - \frac{\pi x}{2}\right) + \frac{\pi x^2}{4}$

$= 500x - \frac{\pi x^2}{2} + \frac{\pi x^2}{4}$

$= \frac{2000x - \pi x^2}{4}$

(3)

(iii) $\frac{dA}{dx} = \frac{1}{4}(2000 - 2\pi x)$

Maximum area $\Rightarrow \frac{dA}{dx} = 0$

$\therefore \frac{1}{4}(2000 - 2\pi x) = 0$

$\therefore 2\pi x = 2000$

$x = \frac{2000}{2\pi} = \frac{1000}{\pi}$

$\frac{d^2A}{dx^2} = -2\pi < 0$

$\therefore$ maximum area at $x = \frac{1000}{\pi}$.

$\therefore$ Max. Area $= \frac{1}{4}\left[2000\cdot\frac{1000}{\pi} - \pi\frac{(1000)^2}{\pi^2}\right]$

$= \frac{1}{4}\left[1000000\left(\frac{2}{\pi} - \frac{1}{\pi}\right)\right]$

$= \frac{25}{\pi} = 7.95 \doteq 8$ Hectares

(3)

Question 5

(a)

(i) $\int \frac{dx}{x^5} = \int x^{-5}\,dx = \frac{x^{-4}}{-4} + c$

$= \frac{-x^{-4}}{4} + c$

$= -\frac{1}{4x^4} + c$

(1)

(ii) $\int x^2(5 - 3x)\,dx = \int (5x^2 - 3x^3)\,dx$

$= \frac{5\cdot x^3}{3} - \frac{3x^4}{4} + c$

(2)

(iii) $\int \frac{dx}{\sqrt{9-2x}} = \int (9-2x)^{-\frac{1}{2}}\,dx$

$= \frac{2(9-2x)^{\frac{1}{2}}}{-2} + c$

$= -\sqrt{9-2x} + c$

(3)

(iv) $\int (3x+1)^4\,dx$

$= \frac{(3x+1)^5}{5\times 3} + c$

$= \frac{(3x+1)^5}{15} + c$

(2)

(v) $\int \frac{2x^3 - x^4}{4x}\,dx$

$= \frac{1}{4}\int (2x^2 - x^3)\,dx$

$= \frac{1}{4}\left[\frac{2\cdot x^3}{3} - \frac{x^4}{4}\right] + C$

$= \frac{x^3}{6} - \frac{x^4}{16} + c$

(3)

(b) (i) $\int_1^9 (1 + \sqrt{x})\,dx$

$= \left[x + \frac{2x^{\frac{3}{2}}}{3}\right]_1^9$

$= \left(9 + \frac{2}{3}\cdot 9^{\frac{3}{2}}\right) - \left(1 + \frac{2}{3}\right)$

$= 9 + 18 - \frac{5}{3} = \frac{76}{3}$

(3)

(ii) $\int_2^3 (2x-5)^3\,dx$

$= \left[\frac{(2x-5)^4}{4\times 2}\right]_2^3$

$= \frac{1}{8}\left[1^4 - (-1)^4\right] = 0$

(3)

(iii) $\int_0^1 \frac{x^3 - 2x^2 + 3x}{x}\,dx$

$= \int_0^1 (x^2 - 2x + 3)\,dx$

$= \left[\frac{x^3}{3} - x^2 + 3x\right]_0^1$

$= \frac{1}{3} - 1 + 3 - 0 = \frac{7}{3}$

(3)

(iv) $\int_1^2 \frac{dx}{(3x-1)^2}$

$= \int_1^2 (3x-1)^{-2}\,dx$

$= \left[\frac{(3x-1)^{-1}}{-3}\right]_1^2 = -\frac{1}{3}\left[\frac{1}{3x-1}\right]_1^2$

$= -\frac{1}{3}\left(\frac{1}{5} - \frac{1}{2}\right) = \frac{1}{10}$

(3)

Question 6

a) $\frac{dy}{dx} = 3x^2 - 2x + 1$

$\therefore y = \int (3x^2 - 2x + 1)\,dx + c$

$= x^3 - x^2 + x + c$

passes through $(2,3)$

$\Rightarrow 3 = 2^3 - 2^2 + 2 + c$

$\therefore c = -3$

$\therefore y = x^3 - x^2 + x - 3$

(3)

b) $\int_{-2}^{5} f(x)\,dx$ = Area A − Area B + Area C

$= 8 - 3 + 1 = 6u^2$

c) (i) $y = x^3 + x^2 - x + 2$

$\frac{dy}{dx} = 3x^2 + 2x - 1$

$= (3x-1)(x+1)$

Turning points occur when $\frac{dy}{dx} = 0$.

$\therefore (3x-1)(x+1) = 0$

$\therefore x = \frac{1}{3}$ or -1

when $x = \frac{1}{3}$, $y = \frac{49}{27}$

when $x = -1$, $y = 3$

$\therefore$ co-ordinates of $A = (-1, 3)$

$B = \left(\frac{1}{3}, \frac{49}{27}\right)$

(ii) $\frac{d^2y}{dx^2} = 6x + 2$

Concave up $\Rightarrow \frac{d^2y}{dx^2} > 0$

$\therefore 6x + 2 > 0$

$\therefore x > -\frac{1}{3}$.