GIRRAWEEN HIGH SCHOOL

Task 2

2014

MATHEMATICS

Time allowed – 90 minutes
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL 6 questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are on laminated sheet provided.
- Board-approved calculators may be used.
- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1 , Question 2 , etc. Each piece of paper must show your name.
- You may ask for extra pieces of paper if you need them.

FINAL MARK

GIRRAWEEN HIGH SCHOOL
MATHEMATICS
2014

HIGHER SCHOOL CERTIFICATE EXAMINATION

ANSWERS COVER SHEET

Name: ______________________________

QUESTION	MARK	H2	H3	H4	H5	H6	H7	H8	H9
Section 1	/5								√
Part B Q6	/10				√				√
Q 7	/14				√	√			√
Q 8	/10				√	√			√
Q 9	/11				√	√			√
Q 10	/22				√	√			√
Q 11	/9				√	√		√	√
TOTAL	/81				/76	/66		/9	/81

HSC Outcomes **Mathematics**

H1 seeks to apply mathematical techniques to problems in a wide range of practical contexts.

H2 constructs arguments to prove and justify results.

H3 manipulates algebraic expressions involving logarithmic and exponential functions.

H4 expresses practical problems in mathematical terms based on simple given models.

H5 applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems.

H6 uses the derivative to determine the features of the graph of a function.

H7 uses the features of a graph to deduce information about the derivative.

H8 uses techniques of integration to calculate areas and volumes.

H9 communicates using mathematical language, notation, diagrams and graphs.

GIRRAWEEN HIGH SCHOOL
MATHEMATICS

YEAR 12 HSC **Task 2 2014**

Time Allowed: 90 minutes

INSTRUCTIONS TO STUDENTS

- Attempt **ALL 6** questions.
- Circle the best response for the questions in **Part A**
- Start each question in **Part B** on a new page.
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.

PART A (5 marks)

For questions 1-5 circle the best response from the following:

Question1. What are the x-coordinates of the two stationary points to the curve $y=5+3x^3-2x^4$?

(A) $x=0,\ x=\frac{2}{3}$

(B) $x=0,\ x=\frac{3}{2}$

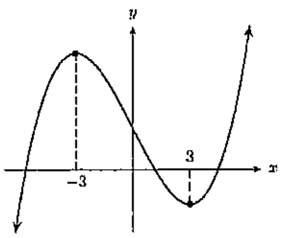
(C) $x=0,\ x=\frac{8}{9}$

(D) $x=0,\ x=\frac{9}{8}$

Question 2. For which values of x is the curve $f(x)=-x^2+4x+5$ increasing?

(A) $x>2$ (B) $x<2$ (C) $x\geq -2$ (D) $x\leq -2$

Question 3. From the graph of $y=f(x)$, when is $f'(x)$ negative?

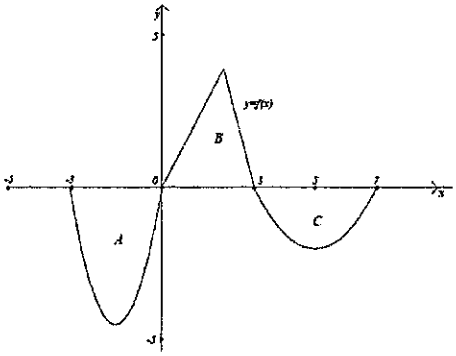


(A) $x<-3$ or $x>3$ (B) $-3<x<3$ (C) $x\leq -3$ or $x\geq 3$ (D) $-3\leq x\leq 3$

Question 4. What is the value of $\int_3^5 (x+1)^{-2}dx$?

(A) $\frac{1}{12}$ (B) $\frac{1}{24}$ (C) $\frac{5}{24}$ (D $-\frac{5}{24}$

Question 5.



The graph on the right shows the curve $y = f(x)$. The shaded areas are bounded by $y = f(x)$ and the x axis. Shaded area A is 9 square units, shaded area B is 6 square units and shaded area C is 5 square units.

The value of $\int_{-3}^{7} f(x)dx$

(A) 8 (B) -8 (C) 20 (D) -20

PART B

QUESTION 6. (10 marks)

(a) Show that the curve $y = 2x - \frac{3}{x-1}$ is increasing for all values of x . **2**

(b) The curve $y = x^3 + ax^2 + bx + 5$ has a stationary point at (2,-3). Find the values of a and b . **3**

(c) If $y = \frac{x}{x-1}$, show that $\frac{2yy'}{x} + y'' = 0$. **3**

(d) For what values of x is the curve $y = 2x^3 + 3x^2 - 12x + 8$ concave up? **2**

QUESTION 7 (14 marks)

(a) Show that the curve $y = x^4 - x + 1$ has no points of inflexion. 3

(b) The function $y = x^3 - 3x^2 - 9x + 1$ is defined in the domain $-4 \le x \le 5$.

(i) Find the coordinates of any turning points and determine their nature. 4

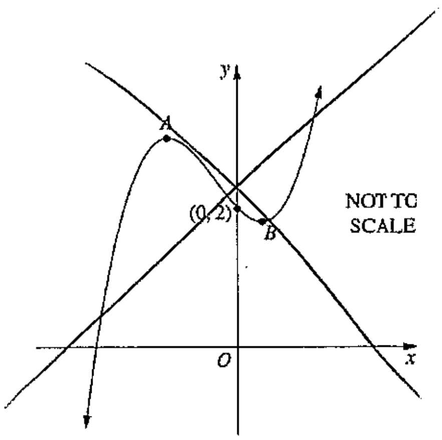
(ii) Find the coordinates of any points of inflexion. 2

(iii) Draw a neat sketch of the curve. 3

(iv) Determine the minimum value of the function y, in the domain $-4 \le x \le 5$ 2

Question 8. (10 marks)

(a)



~~The diagram shows a sketch of the~~ curve $y = 6x^2 - x^3$. The curve cuts the x axis at L, and has a local maximum at M and a point of inflection at N.

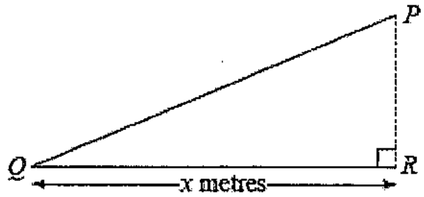
(i) Find the coordinates of L. 1

(ii) Find the coordinates of M. 1

(iii) Find the coordinates of N. 1

Question 8 continued

(b)



A wire of length 5 metres is to be bent to form the hypotenuse and base of a right –angled triangle PQR, as shown in the diagram. Let the length of the base QR be x metres.

(i) What is the length of the hypotenuse PQ in terms of x? **1**

(ii) Show that the area of the triangle PQR is $\frac{1}{2}x\sqrt{25-10x}$ square metres. **2**

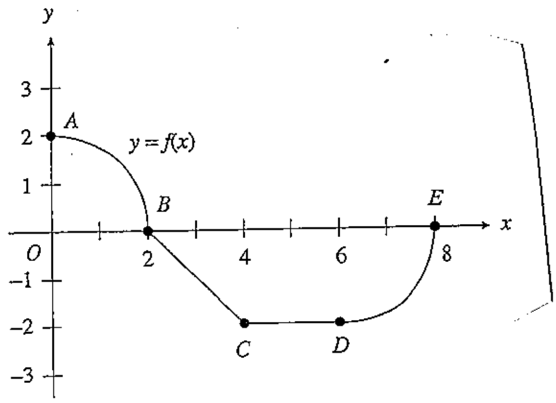
(iii) What is the maximum possible area of the triangle? **4**

QUESTION 9. (11marks)

(a) Find the primitive function of $\frac{2}{\sqrt{x}}+\frac{1}{\sqrt[3]{x^2}}$. **3**

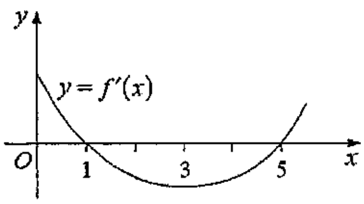
(b) Find $f(x)$, given that $f'(x)=6x^2-6x+5$ and that the curve passes through the point (2,13). **3**

(c)



The graph of the function f consists of a quarter circle AB, a straight line segment BC, a horizontal straight line segment CD, and a quarter circle DE as shown above. Evaluate $\int_0^8 f(x)dx$. **3**

(d)



The diagram shows the graph of the gradient function of the curve $y=f(x)$. For what values of x does $f(x)$ have a local minimum? Justify your answer. **2**

Question 10.(22 marks)

(a) Find each indefinite integral:

(i) $\int (3x^2 - 2)dx$ **1**

(ii) $\int (\sqrt{x} - 2)(\sqrt{x} + 2)dx$ **3**

(iii) $\int \frac{dx}{(2x+3)^2}$ **3**

(iv) $\int \frac{\sqrt{x} - 1}{2\sqrt{x}}dx$ **3**

(b) Evaluate:

(i) $\int_0^1 (3x^6 + 1)dx$ **2**

(ii) $\int_0^3 \frac{dx}{(2x-3)^2}$ **3**

(iii) $\int_2^7 \frac{dx}{\sqrt{x+2}}$ **3**

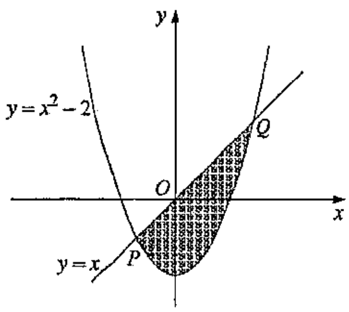
(iv) (i) Find $\frac{d}{dx}(x^2 + 1)^5$. **2**

(ii) Hence find $\int 10x(x^2 + 1)^4 dx$ **2**

Question 11.(9 marks)

(a) Calculate the area enclosed by the curve $y = 4x^2$, the y axis and the lines $y = 1$, $y = -4$, lying in the first quadrant. **4**

(b) The diagram shows the graphs of $y = x^2 - 2$ and $y = x$.



(i) Find the x values of the points of intersection, P and Q. **2**

(ii) Calculate the area of the shaded region. **3**

End of Examination

Year 12 Mathematics Task 2 2014 Solutions.

Part A

① D ③ B ⑤ B
② B ④ A

Part B

Question 6

a) $y = 2x - \frac{3}{x-1}$, $x \neq 1$

$y' = 2 + 3(x-1)^{-2}$
$= 2 + \frac{3}{(x-1)^2}$

$(x-1)^2 \geq 0$ for all values of x.
$\therefore y' > 0$ for all values of x.
$\therefore$ The curve is increasing for all values of x

(2)

b) $y = x^3 + ax^2 + bx + 5$

$y' = 3x^2 + 2ax + b$.

Since $(2, -3)$ is a stationary point, when $x = 2$, $y' = 0$

$\therefore 0 = 12 + 4a + b$

$\therefore 4a + b = -12$ — ①

Since the curve passes through $(2, -3)$, in $y = x^3 + ax^2 + bx + 5$,

$\left.\begin{matrix} x=2 \\ y=-3 \end{matrix}\right\}$ $-3 = 8 + 4a + 2b + 5$

$\therefore 4a + 2b = -16$ — ②

$4a + b = -12$ — ①
$4a + 2b = -16$ — ②

① − ② $-b = 4$ $\therefore b = -4$

$\therefore a = -2$

(3)

c) $y = \frac{x}{x-1}$

$y' = \frac{1(x-1) - x}{(x-1)^2} = \frac{-1}{(x-1)^2}$

$y'' = 2(x-1)^{-3} = \frac{2}{(x-1)^3}$

LHS $\frac{2yy'}{x} + y'' =$

$= \frac{2}{x} \times \frac{x}{x-1} \times \frac{-1}{(x-1)^2} + \frac{2}{(x-1)^3}$

$= 0 = \text{RHS}$

(3)

d) $y = 2x^3 + 3x^2 - 12x + 8$

The curve is concave up $\Rightarrow y'' > 0$

$y' = 6x^2 + 6x - 12$
$y'' = 12x + 6 > 0 \Rightarrow x > -\frac{1}{2}$

(2)

Question 7.

a) $y = x^4 - x + 1$

$y' = 4x^3 - 1$

$y'' = 12x^2$

At points of inflexion, $y'' = 0$

$\therefore 12x^2 = 0 \quad \therefore x = 0$

when $x = 0$, $y = 1$

$\therefore (0,1)$ is a possible point of inflexion.

Concavity test

x	-0.1	0	0.1
y''	$+$	0	$+$

There is no change of concavity.

$\therefore (0,1)$ is not a p.o.inflexion.

$\therefore$ The curve has no p.o.inflexion. (3)

b) $y = x^3 - 3x^2 - 9x + 1$

$y' = 3x^2 - 6x - 9$

At stationary points $y' = 0$

$\therefore 3(x^2 - 2x - 3) = 0$

$3(x-3)(x+1) = 0$

$\therefore$ stationary points at $x = -1, 3$

When $x = -1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 1$
$= 6$

$x = 3$, $y = -26$

$y'' = 6x - 6$

at $x = -1$, $y'' < 0 \therefore$ maximum

at $x = 3$, $y'' > 0 \therefore$ minimum

$\therefore$ maximum at $(-1, 6)$

minimum at $(3, -26)$ (4)

(ii) For possible points of inflexion

$y'' = 0 \Rightarrow 6(x-1) = 0$

$\therefore x = 1$

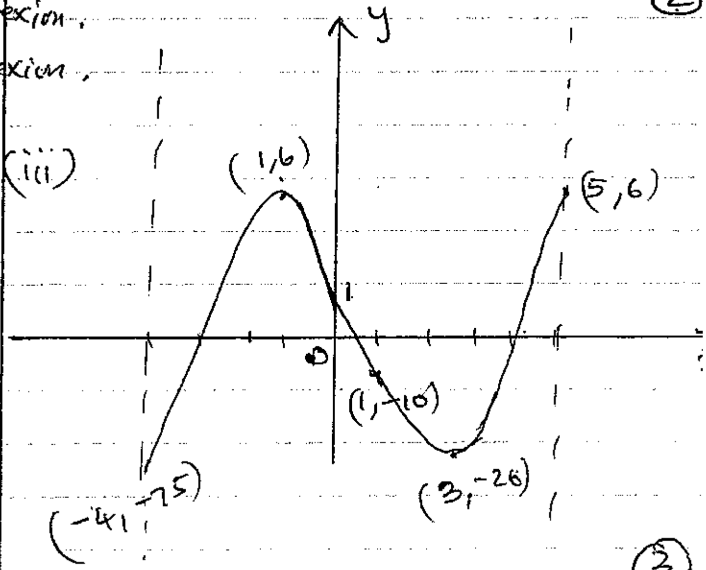
Concavity test:

x	0	1	2
y''	$-$	0	$+$

$\therefore$ Concavity change occurs when $x = 1$, $y = -10$

$\therefore (1, -10)$ is a point of infl (2)

(iii)



(3)

(iv) minimum value $= -7$

(2)

Question 8

a) (i) $y = 6x^2 - x^3$
$= x^2(6 - x)$

At L, $y = 0$
$\therefore x = 0$ or 6

$\therefore$ coordinates of L are (6,0) (1)

(ii) $\frac{dy}{dx} = 12x - 3x^2$
$= 3x(4 - x)$

at M, $\frac{dy}{dx} = 0$

$\therefore 3x(4-x) = 0$
$\therefore x = 0$ or 4
when $x = 4$, $y = 6 \cdot 4^2 - 4^3$
$= 32$

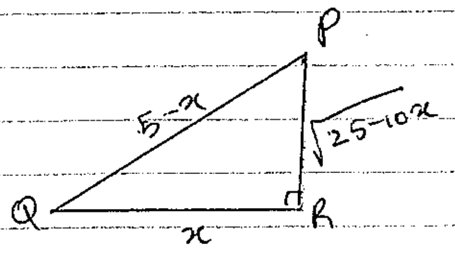
$\therefore N = (4, 32)$ (1)

(iii) $\frac{d^2y}{dx^2} = 12 - 6x = 6(2 - x)$

At N, $\frac{d^2y}{dx^2} = 0 \Rightarrow x = 2$

$\therefore N = (2, 16)$ (1)

b)



(i) $PQ = (5 - x)$m

(ii) $PR^2 = (5 - x)^2 - x^2$
$= 25 - 10x$
$\therefore PR = \sqrt{25 - 10x}$, $(PR > 0)$

(1)

$\therefore$ Area of ΔPQR
$= \frac{1}{2} \times b \times h$
$= \frac{1}{2} \times x \times \sqrt{25 - 10x}$
$= \frac{1}{2} x \sqrt{25 - 10x}$ m²

(2)

(iii) $A = \frac{x}{2} \sqrt{25 - 10x}$

$u = \frac{1}{2}x$ $\quad v = (25 - 10x)^{\frac{1}{2}}$
$u' = \frac{1}{2}$ $\quad v' = \frac{1}{2}(25 - 10x)^{-\frac{1}{2}} \times -10$
$v' = -5(25 - 10x)^{-\frac{1}{2}}$
$v' = \frac{-5}{\sqrt{25 - 10x}}$

$y' = u'v + uv'$
$= \frac{\sqrt{25 - 10x}}{2} - \frac{5x}{2\sqrt{25 - 10x}}$
$= \frac{25 - 10x - 5x}{2\sqrt{25 - 10x}}$
$= \frac{25 - 15x}{2\sqrt{25 - 10x}}$

At stationary points, $\frac{dA}{dx} = 0$

$\therefore \frac{-15x + 25}{2\sqrt{25 - 10x}} = 0$

$\therefore 15x = 25 \Rightarrow x = \frac{5}{3}$

Concavity test

x	1	$\frac{5}{3}$	2
$\frac{dA}{dx}$	+	0	−

(4)

$\therefore$ Concavity changes
$\therefore$ A maximum occurs at
$x = \frac{5}{3}$

$\therefore A_{max} = \frac{1}{2} \cdot \frac{5}{3} \cdot \sqrt{25 - 10 \cdot \frac{5}{3}} = \frac{25\sqrt{3}}{18}$ m²

Question 9

a) $f'(x) = \dfrac{2}{\sqrt{x}} + \dfrac{1}{\sqrt[3]{x^2}}$

$= 2x^{-\frac{1}{2}} + x^{-\frac{2}{3}}$

$\therefore f(x) = \dfrac{2x^{\frac{1}{2}}}{\frac{1}{2}} + \dfrac{x^{\frac{1}{3}}}{\frac{1}{3}} + c$

$= 4x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + c$

$= 4\sqrt{x} + 3\sqrt[3]{x} + c$

(3)

b) $f'(x) = 6x^2 - 6x + 5$

$f(x) = 6 \cdot \dfrac{x^3}{3} - 6\dfrac{x^2}{2} + 5x + c$

$= 2x^3 - 3x^2 + 5x + c$

sub in $f(2) = 13 \Rightarrow$

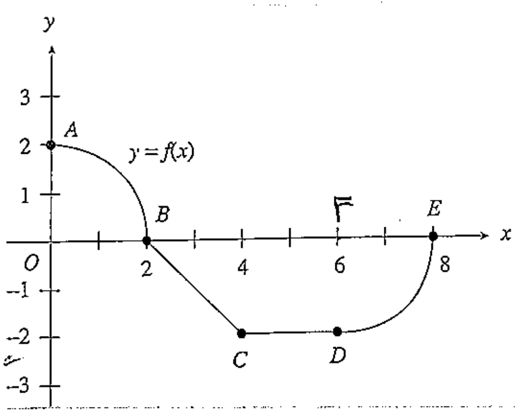
$13 = 16 - 12 + 10 + c$

$\therefore c = -1$

$\therefore f(x) = 2x^3 - 3x^2 + 5x - 1$

(3)

c)



Let $F = (6,0)$

$\therefore \int_0^8 f(x)\,dx = ($area $\frac{1}{4}$ circle AB

− Area of $\frac{1}{4}$ circle DE

− area of trapezium BCDF)

$= 0 - \frac{1}{2}(2+4) \cdot 2 = -6$

(3)

d) At $x = 5$, $f'(x) = 0$

when $x < 5$, $f'(x) < 0$

when $x > 5$, $f'(x) > 0$

$\therefore$ A local ~~maximum~~ minimum at $x = 5$

(2)

Question 10

a) (i) $\int (3x^2 - 2)\,dx = x^3 - 2x + c$

(1)

(ii) $\int (\sqrt{x} - 2)(\sqrt{x} + 2)\,dx$

$= \int (x - 4)\,dx = \dfrac{x^2}{2} - 4x + c$

(2)

(iii) $\int \dfrac{dx}{(2x+3)^2} = \int (2x+3)^{-2}\,dx$

$= \dfrac{(2x+3)^{-1}}{-1 \times 2} + c$

$= -\dfrac{1}{2(2x+3)} + c$

(3)

(iv) $\int \dfrac{\sqrt{x} - 1}{2\sqrt{x}}\,dx$

$= \int \dfrac{1}{2} - \dfrac{1}{2\sqrt{x}}\,dx$

$= \dfrac{x}{2} - \dfrac{1}{2} \cdot \dfrac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \dfrac{x}{2} - \sqrt{x} + c$

Question 10

a) (i) $\int (3x^2-2)\,dx$

$= x^3 - 2x + c$ (1)

(ii) $\int (\sqrt{x}-2)(\sqrt{x}+2)\,dx$

$= \int (x-4)\,dx = \frac{x^2}{2} - 4x + c$

(2)

(iii) $\int \frac{dx}{(2x+3)^2} = \int (2x+3)^{-2}\,dx$

$= -\frac{1}{2(2x+3)} + c$

(iv) $\int \frac{\sqrt{x}-1}{2\sqrt{x}} = \int \left(\frac{1}{2} - \frac{1}{2\sqrt{x}}\right) dx$

$= \frac{x}{2} - \frac{2x^{\frac{1}{2}}}{2} + c$

$= \frac{x}{2} - \sqrt{x} + c$

(3)

b) (i) $\int_0^1 (3x^6+1)\,dx = \left[\frac{3x^7}{7} + x\right]_0^1$

$= 1\frac{3}{7}$ (2)

(ii) $\int_0^3 \frac{dx}{(2x-3)^2} = \left[\frac{(2x-3)^{-1}}{-2}\right]_0^3$

$= -\frac{1}{2}\left[\frac{1}{2x-3}\right]_0^3 = -\frac{1}{2}\left[\frac{1}{3} + \frac{1}{3}\right]$

$= -\frac{1}{3}$ (3)

(ii) $\int_2^7 \frac{dx}{\sqrt{x+2}} = \int_2^7 (x+2)^{-\frac{1}{2}}\,dx$

$= \left[\frac{(x+2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^7 = 2\left[\sqrt{x+2}\right]_2^7$

$= 2\left[3-2\right] = 2$

(3)

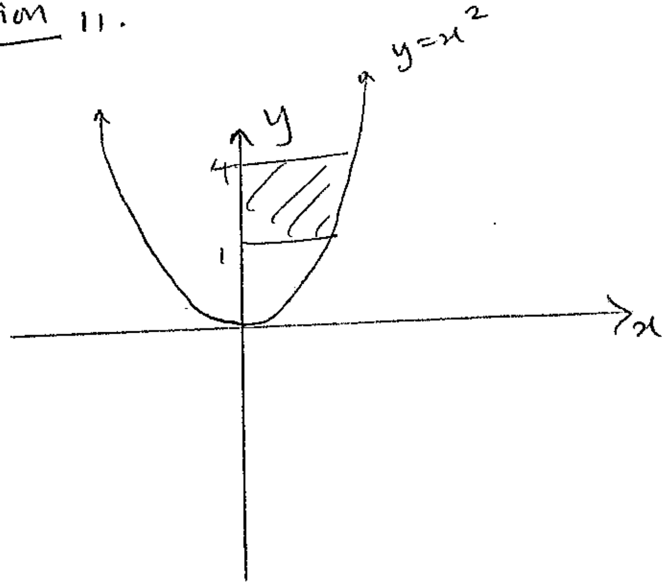
(iv) (α) $\frac{d}{dx}(x^2+1)^5 = 5(x^2+1)^4 \times 2x$

$= 10x(x^2+1)^4$

(β) $\therefore \int 10x(x^2+1)^4\,dx = (x^2+1)^5 + c$

Question 11.

a)



$y = 4x^2 \quad \therefore \; x = \frac{\sqrt{y}}{2}$

$A = \int_1^4 x\,dy = \int_1^4 \frac{\sqrt{y}}{2}\,dy$

$= \left[\frac{y^{\frac{3}{2}}}{3}\right]_1^4 = \frac{7}{3}u^2.$

(4)

b) (i) At points of intersection,

$x^2 - 2 = x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\therefore x = -1, 2$ (2)

(ii) Area of shaded region.

$= \int_{-1}^{2} x - (x^2 - 2)\,dx$

$= \int_{-1}^{2} (x - x^2 + 2)\,dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2}$

$= \left[\frac{4}{2} - \frac{8}{3} + 4\right] - \left[\frac{1}{2} + \frac{1}{3} - 2\right] = 4\frac{1}{2}u^2$

Question 10

a) (i) $\int (3x^2 - 2)\,dx$

$= x^3 - 2x + c$ (1)

(ii) $\int (\sqrt{x} - 2)(\sqrt{x} + 2)\,dx$

$= \int (x - 4)\,dx = \frac{x^2}{2} - 4x + c$ (2)

(iii) $\int \frac{dx}{(2x+3)^2} = \int (2x+3)^{-2}\,dx$

$= \frac{-1}{2(2x+3)} + c$

(iv) $\int \frac{\sqrt{x} - 1}{2\sqrt{x}} = \int \left(\frac{1}{2} - \frac{1}{2\sqrt{x}}\right) dx$

$= \frac{x}{2} - \frac{2x^{\frac{1}{2}}}{2} + c$

$= \frac{x}{2} - \sqrt{x} + c$ (3)

b) (i) $\int_0^1 (3x^6 + 1)\,dx = \left[\frac{3x^7}{7} + x\right]_0^1$

$= 1\frac{3}{7}$ (2)

(ii) $\int_0^3 \frac{dx}{(2x-3)^2} = \left[\frac{(2x-3)^{-1}}{-2}\right]_0^3$

$= -\frac{1}{2}\left[\frac{1}{2x-3}\right]_0^3 = -\frac{1}{2}\left[\frac{1}{3} + \frac{1}{3}\right]$

$= -\frac{1}{3}$ (3)

(ii) $\int_2^7 \frac{dx}{\sqrt{x+2}} = \int_2^7 (x+2)^{-\frac{1}{2}}\,dx$

$= \left[\frac{(x+2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^7 = 2\left[\sqrt{x+2}\right]_2^7$

$= 2\left[3 - 2\right] = 2$

(3)

(iv) (α) $\frac{d}{dx}(x^2+1)^5 = 5(x^2+1)^4 \times 2x$

$= 10x(x^2+1)^4$

(β) $\therefore \int 10x(x^2+1)^4\,dx = (x^2+1)^5 + c$