Girraween High School

Year 12 HSC Task 2 Examination

Mathematics

February 2016

Time Allowed: 90 Minutes

Instructions:

- There are 12 questions in this paper. All questions are compulsory.
- Start each question (6 – 12) on a new sheet of paper.
- Write on one side of the paper only.
- Show all necessary working.
- Board-approved calculators may be used.
- Marks may be deducted for careless or badly arranged work.

Section I

5 marks

Attempt Questions 1-5

Question 1 (1 mark)

For which values of x is $f(x) = \dfrac{x^4}{12} - \dfrac{x^2}{2}$ concave up?

A. $x > 1$

B. $x < -1$

C. $-1 < x < 1$

D. $x < -1$ and $x > 1$

Question 2 (1 mark)

A function $f(x)$ is increasing at an increasing rate at $x = a$, which of the following is true?

A. $f'(a) < 0$ and $f''(a) < 0$

B. $f'(a) < 0$ and $f''(a) > 0$

C. $f'(a) > 0$ and $f''(a) < 0$

D. $f'(a) > 0$ and $f''(a) > 0$

Question 3 (1 mark)

Which of the following is the correct value of $\displaystyle\int_0^1 (3x+1)^3\,dx$?

A. $\dfrac{64}{3}$

B. $\dfrac{85}{4}$

C. $\dfrac{255}{4}$

D. 64

Question 4 (1 mark)

Suppose $b > a$, which of the following is always greater than $\displaystyle\int_a^b f(x)\,dx$?

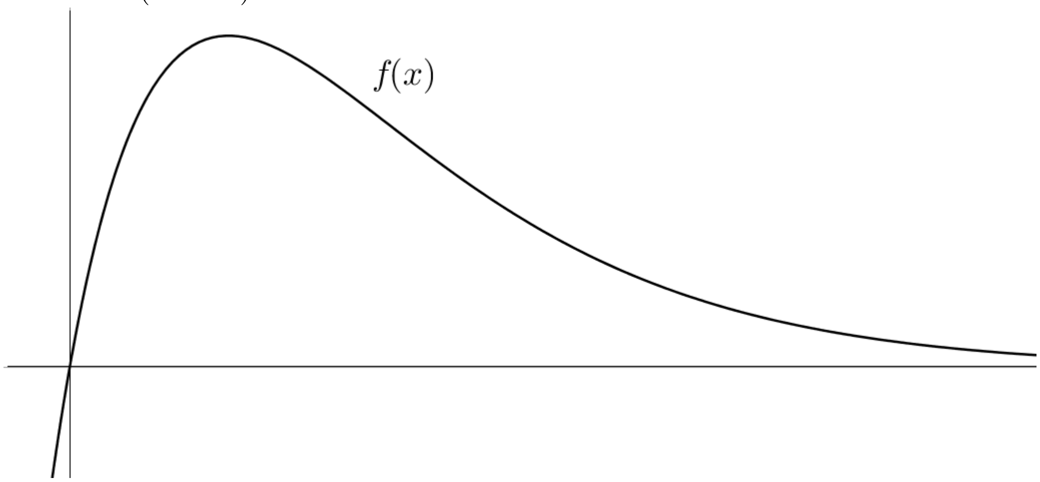
A. $\displaystyle\int_a^b kf(x)\,dx$ where k is a constant such that $k > 0$

B. $\displaystyle\int_a^b f(x) + k\,dx$ where k is a constant such that $k > 0$

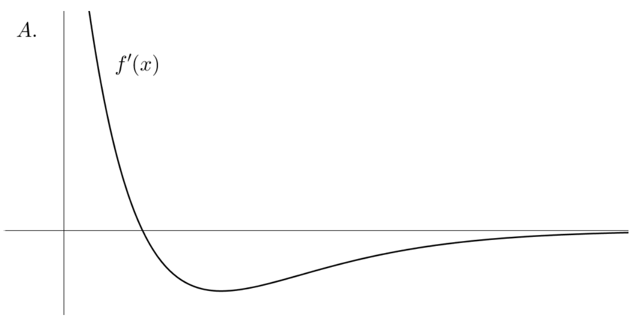
C. $\displaystyle\int_a^b f^2(x)\,dx$

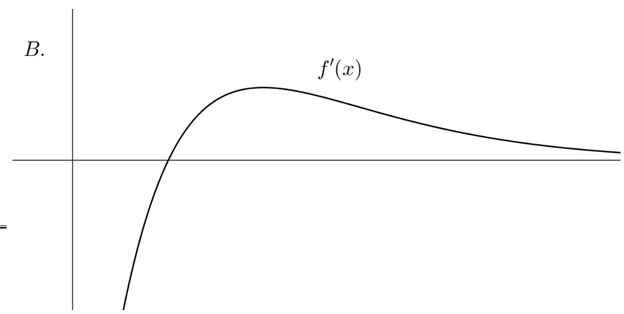
D. $\left|\displaystyle\int_a^b f(x)\,dx\right|$

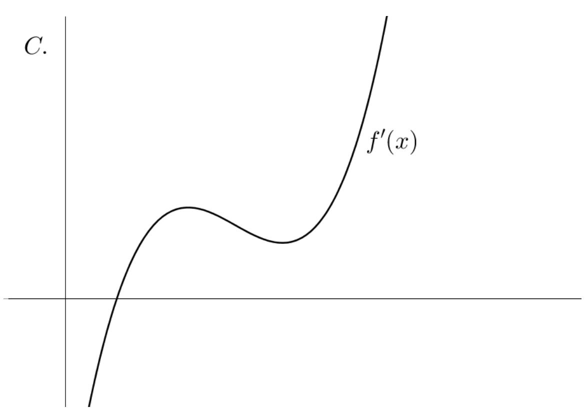
Question 5 (1 mark)

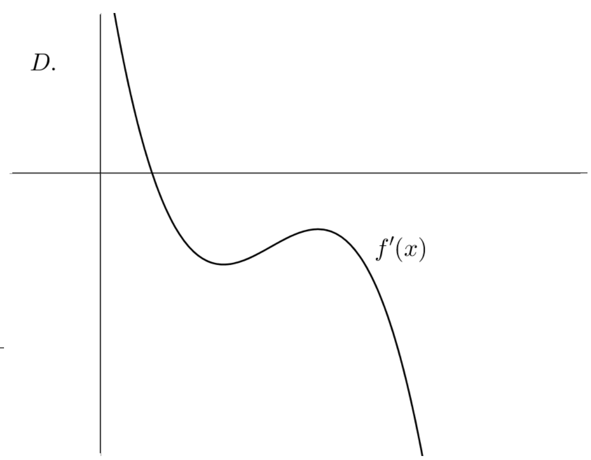


A graph of $f(x)$ is given above. Which of the following best represents the graph of $f'(x)$?









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Section II

71 marks

Attempt Questions 6-12

Write your answers on the paper provided.

In Questions 6-12, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (8 marks)

(a) The curve $y = \sqrt{2ax^2 + x}$ has a stationary point at $x = \dfrac{1}{2}$. Find the values of a. [3]

(b) If $y = x^2\sqrt{x}$, show that $\dfrac{3y'}{y''} = 2x$ [3]

(c) Find the values of x such that $y = x(x+1)^3$ is increasing. [2]

Question 7 (12 marks)

(a) Show that the curve $y = \dfrac{x}{2x+1}$ does not have any points of inflexion. [3]

(b) Consider the function $f(x) = 4x^3 - x^4$.

i. Find the coordinates of the stationary points of the curve $f(x)$, and determine their nature. [4]

ii. Find any points of inflexion. [2]

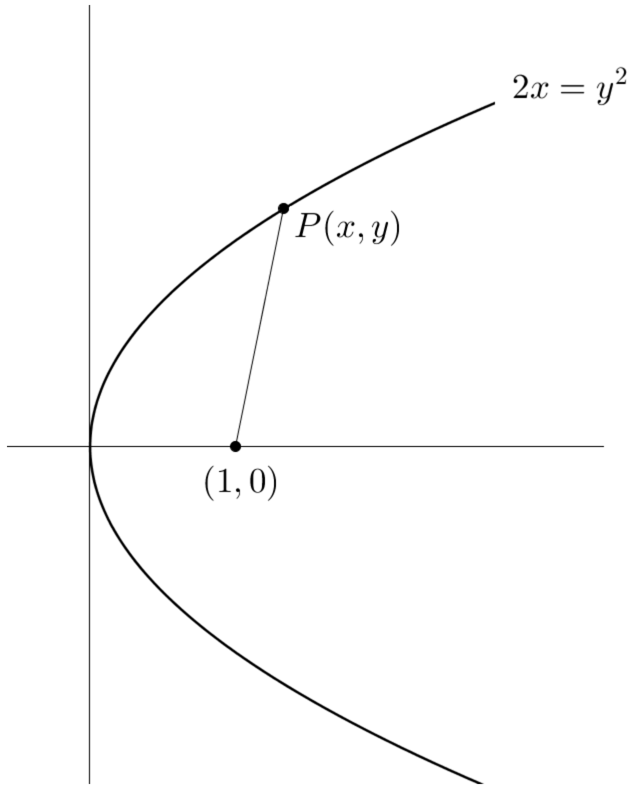
iii. Sketch the graph of $f(x)$, clearly indicating all important features including intercepts. [3]

The exam continues on the next page

Question 8 (8 marks)

(a) Determine the global minimum value of the function $f(x) = x^4 - 4x^2 + 3$ in the domain $-1 \leq x \leq 2$ [3]

(b) Suppose point $P(x, y)$ is a point on the parabola $2x = y^2$.



i. Show that the distance D between $P(x, y)$ and $(1, 0)$ is given by $D = \sqrt{x^2 + 1}$ [2]

ii. Using calculus, find the coordinates of $P(x, y)$ such that D is minimum. [3]

Question 9 (10 marks)

(a) Find the primitive of the following:

i. $f(x) = 2\sqrt{x} - \dfrac{2}{\sqrt[3]{x}} + 1$ [2]

ii. $f(x) = \dfrac{1}{\sqrt{2x + 1}}$ [2]

(b) The gradient function of a curve is given by $y' = 3(x + 1)^2$. Find the equation of the curve if it passes through the point $(1, 10)$. [2]

(c) The second derivative of a curve is given by $\dfrac{d^2y}{dx^2} = 1 + \dfrac{1}{x^3}$. Find the equation of the curve given that it passes through the points $(1, 0)$ and $(-1, 5)$. [4]

The exam continues on the next page

Question 10 (12 marks)

(a) Find:

i. $\displaystyle\int x^3 + 4x^2 - 1\,dx$ [1]

ii. $\displaystyle\int 6(1-2x)^5\,dx$ [2]

iii. $\displaystyle\int \sqrt{x}\left(x+\frac{1}{x}\right)\,dx$ [2]

(b) Evaluate:

i. $\displaystyle\int_{-2}^{2} x^5 - x^3 + x\,dx$ [2]

ii. $\displaystyle\int_{0}^{10} \frac{3}{10\left(\frac{x}{5}-1\right)^4}\,dx$ [2]

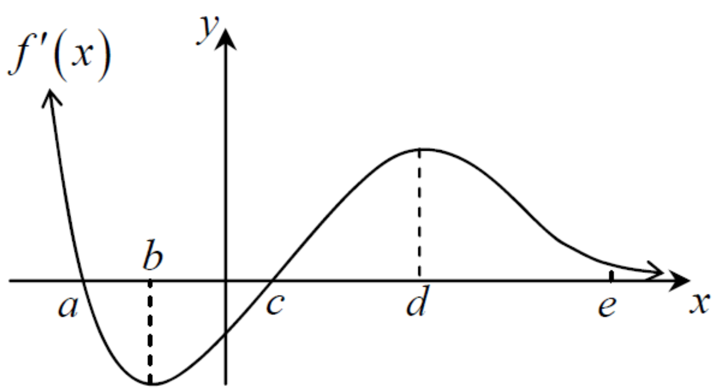
(c) Find the positive value of k such that $\displaystyle\int_{1}^{k} k + x\,dx = 0$ [3]

Question 11 (10 marks)

(a) Find the area bounded by the curve $y = x^2 - 4x + 3$ and the x-axis between $x = 2$ and $x = 4$. [3]

(b) Find the area bounded by the curve $y = \sqrt{x-1}$ and the y-axis between $y = 0$ and $y = 1$ [3]

(c) The graph below represents $y = f'(x)$. Specific x-values a, b, c, d and e are as indicated in the diagram.



i. For what value(s) of x will the graph of $y = f(x)$ have a stationary point? [1]

ii. For what value(s) of x is the graph of $y = f(x)$ increasing? [2]

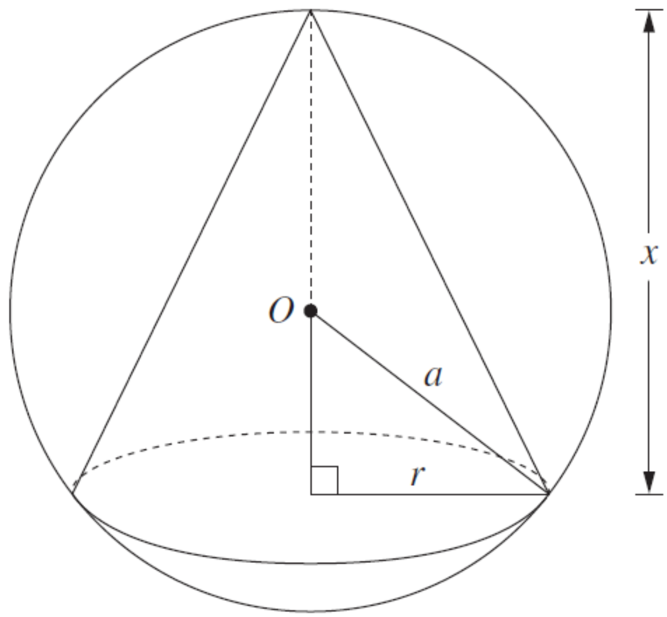
iii. For what value(s) of x is the graph of $f(x)$ concave up? [1]

The exam continues on the next page

Question 12 (11 marks)

(a) If $y = ax^3 + bx^2 + cx + d$ has only one stationary point show that $b^2 = 3ac$. [2]

(b) A cone is inscribed in a sphere of radius a, centred at O. The height of the cone is x and the radius of the base is r, as shown in the diagram below.



i. Show that $r^2 = 2ax - x^2$. [2]

ii. Show that the volume, V, of the cone is given by $V = \dfrac{1}{3}\pi(2ax^2 - x^3)$. [1]

iii. Find the value of x for which the volume of the cone is a maximum. [3]

(c) Find the value of $\displaystyle\int_{-1}^{2} |x| + x^2\, dx$ [3]

End of exam

Yr12 HSC TASK 2 (2U) 2016

Multiple Choice Solutions:

D D B B A

Q6

(a) $y = \sqrt{2ax^2+x}$

$y = (2ax^2+x)^{1/2}$

$y' = \frac{1}{2}(2ax^2+x)^{-1/2} \times (4ax+1)$

$y' = \dfrac{4ax+1}{2\sqrt{2ax^2+x}}$

$y'=0$ when $x=\frac{1}{2}$

$\therefore 4ax+1=0$ when $x=\frac{1}{2}$

$\therefore 2a+1=0 \quad \therefore a=-\frac{1}{2}$.

(b) $y = x^2\sqrt{x}$

$y = x^{5/2}$

$y' = \frac{5}{2}x^{\frac{3}{2}} = \dfrac{5x\sqrt{x}}{2}$

$y'' = \frac{15}{4}x^{\frac{1}{2}} = \dfrac{15\sqrt{x}}{4}$

Required to prove: $\dfrac{3y'}{y''} = 2x$

$LHS = \dfrac{3y'}{y''}$

$= \cancel{3} \times \dfrac{5x\cancel{\sqrt{x}}}{\cancel{2}} \times \dfrac{\cancel{4}}{\cancel{15}\cancel{\sqrt{x}}}$

$= 2x = RHS$.

(c) $y = x(x+1)^3$

$u = x \quad v = (x+1)^3$

$u' = 1 \quad v' = 3(x+1)^2$

$y' = (x+1)^3 + 3x(x+1)^2$

$= (x+1)^2[(x+1)+3x]$

$= (x+1)^2(4x+1)$

Since $(x+1)^2 \geq 0$

$\therefore y' > 0$ when $4x+1 > 0$

$\therefore x > -\frac{1}{4}$

Q7

(a) $y = \dfrac{x}{2x+1}$

$u = x \quad v = 2x+1$

$u' = 1 \quad v' = 2$

$y' = \dfrac{2x+1-2x}{(2x+1)^2}$

$y' = \dfrac{1}{(2x+1)^2}$

$y' = (2x+1)^{-2}$

$y'' = -2(2x+1)^{-3} \times 2$

$y'' = \dfrac{-4}{(2x+1)^3} \neq 0$ for all x.

$\therefore$ no points of inflexion.

Q7

(b)

(i) $f(x) = 4x^3 - x^4 = x^3(4-x)$

$f'(x) = 12x^2 - 4x^3 = 4x^2(3-x)$

$f''(x) = 24x - 12x^2 = 12x(3-x)$

$f'(x) = 0$ when $4x^2(3-x) = 0$

$\therefore\ x = 0\ \&\ x = 3$

$f(0) = 0\ \&\ f(3) = 27$

$\therefore$ Stationary points are:

$(0,0)\ \&\ (3,27)$

$f''(3) = 12(3)(-1) < 0$ $\cap$

$\therefore\ (3,27)$ is max

since $f''(0) = 0$ using $f'(x)$ table to classify:

x	-1	0	1
$f'(x)$	$+$	0	$+$
	/	—	/

$f'(-1) = 4(4) > 0$

$f'(1) = 4(2) > 0$

$\therefore\ (0,0)$ is a horizontal point of inflexion.

(ii)

$f''(x) = 0$ when $12x(2-x) = 0$

$\therefore\ x = 0\ \&\ x = 2.$

$f(2) = 16\ \therefore\ (2,16)$ is a possible point of inflexion.

x	1	2	3
$f''(x)$	$+$	0	$-$

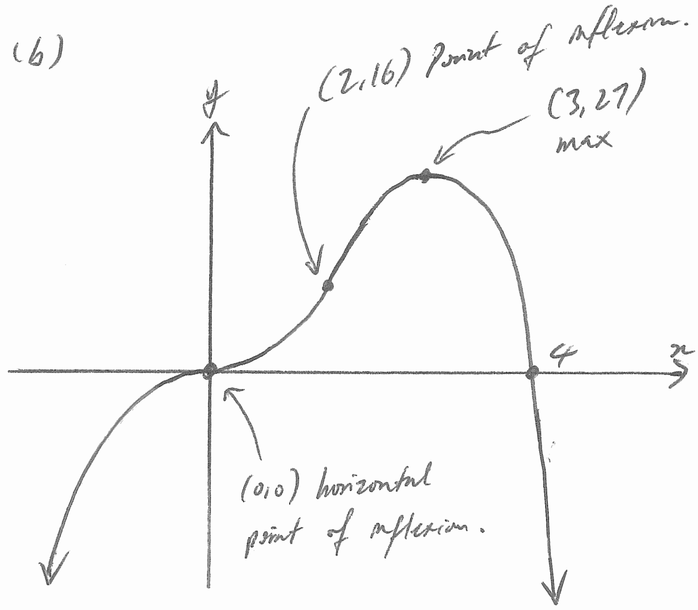
$f''(1) = 12(1) > 0$

$f''(3) = 12(3)(-1) < 0$

$\therefore\ (2,16)$ is a point of inflexion

$(0,0)$ was already classified in part (i).

(b)



Q8

(a) $f(x) = x^4 - 4x^2 + 3$

$f'(x) = 4x^3 - 8x$

$f'(x) = 4x(x^2 - 2)$

$f'(x) = 0$ when $4x(x^2-2) = 0$

when $4x(x-\sqrt{2})(x+\sqrt{2}) = 0$

$\therefore x = 0$ & $x = \pm\sqrt{2}$

$f(-1) = 1 - 4 + 3 = 0$

$f(2) = 2^4 - 2^4 + 3 = 3$

$f(0) = 3$

$f(\sqrt{2}) = 4 - 8 + 3 = -1$

$\therefore$ global minimum value is -1.

Note: $x = -\sqrt{2}$ is ignored as it is outside of the domain.

(b)

(i) $D = \sqrt{(x-1)^2 + y^2}$

but $y^2 = 2x$

$\therefore D = \sqrt{(x-1)^2 + 2x}$

$D = \sqrt{x^2 - 2x + 1 + 2x}$

$D = \sqrt{x^2 + 1}$

(ii)

$D = (x^2+1)^{1/2}$

$\frac{dD}{dx} = \frac{1}{2}(x^2+1)^{-1/2} \times 2x$

$\frac{dD}{dx} = \frac{x}{\sqrt{1+x^2}}$

$\therefore \frac{dD}{dx} = 0$ when $x = 0$.

x	-1	0	1
$\frac{dD}{dx}$	$-$	0	$+$
	\	—	/

$\left.\frac{dD}{dx}\right|_{x=-1} = -\frac{1}{\sqrt{2}}$

$\left.\frac{dD}{dx}\right|_{x=1} = \frac{1}{\sqrt{2}}$

$\therefore x = 0$ gives minimum D.

$y(0) = 0$ $\therefore (0,0)$ is the closest point.

Alternatively:

To minimise D it suffices to minimise D^2

$D^2 = x^2 + 1$

$\frac{dD^2}{dx} = 2x$

$\frac{dD^2}{dx} = 0$ when $x = 0$

$\frac{d^2D^2}{dx^2} = 2 > 0$ $\therefore x = 0$ gives minimum D^2 which gives minimum D.

Q4

(a)

(i) $f(x) = 2\sqrt{x} - \frac{2}{\sqrt[3]{x}} + 1$

$f(x) = 2x^{\frac{1}{2}} - 2x^{-\frac{1}{3}} + 1$

$F(x) = \frac{4}{3}x^{\frac{3}{2}} - 3x^{\frac{2}{3}} + x + C$

$F(x) = \frac{4}{3}x\sqrt{x} - 3x^{\frac{2}{3}} + x + C.$

(ii) $f(x) = \frac{1}{\sqrt{2x+1}}$

$f(x) = (2x+1)^{-\frac{1}{2}}$

$F(x) = \frac{(2x+1)^{\frac{1}{2}}}{2(\frac{1}{2})} + C$

$F(x) = \sqrt{2x+1} + C.$

(b)

$y' = 3(x+1)^2$

$y = 3 \times \frac{(x+1)^3}{3} + C$

$y = (x+1)^3 + C$

but $y(1) = 10$

$\therefore 10 = 2^3 + C \quad \therefore C = 2$

$\therefore y = (x+1)^3 + 2$

(c)

$y'' = 1 + \frac{1}{x^3}$

$y'' = 1 + x^{-3}$

$y' = x - \frac{1}{2}x^{-2} + C$

$y = \frac{x^2}{2} + \frac{1}{2}x^{-1} + Cx + D$

$y = \frac{1}{2}x^2 + \frac{1}{2x} + Cx + D$

$y(1) = 0$

$\therefore 0 = \frac{1}{2} + \frac{1}{2} + C + D$

$\therefore C + D = -1 \quad \cdots ①$

$y(-1) = 5$

$\therefore 5 = \frac{1}{2} - \frac{1}{2} - C + D$

$\therefore -C + D = 5 \quad \cdots ②$

① + ②:

$2D = 4$

$\therefore D = 2$

$\therefore C = -3$

$\therefore y = \frac{1}{2}x^2 + \frac{1}{2x} - 3x + 2$

Q10

(a)

(i) $\int x^3 + 4x^2 - 1 \, dx$

$= \frac{x^4}{4} + \frac{4}{3}x^3 - x + c$

(ii) $\int 6(1-2x)^5 \, dx$

$= 6 \int (1-2x)^5 \, dx$

$= 6 \times \frac{(1-2x)^6}{-2\times 6} + c$

$= -\frac{1}{2}(1-2x)^6 + c$

(iii) $\int \sqrt{x}\left(x + \frac{1}{x}\right) dx$

$= \int x^{\frac{3}{2}} + x^{-\frac{1}{2}} \, dx$

$= \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + c$

$= \frac{2}{5}x^2\sqrt{x} + 2\sqrt{x} + c.$

(b)

(i) $\int_{-2}^{2} x^5 - x^3 + x \, dx = 0$

as $f(x) = x^5 - x^3 + x$ is an odd function.

(b)

(ii) $\int_0^{10} \frac{3}{10\left(\frac{x}{5}-1\right)^4} \, dx$

$= \frac{3}{10} \int \left(\frac{x}{5}-1\right)^{-4} dx$

$= \frac{3}{10} \times \left[\frac{\left(\frac{x}{5}-1\right)^{-3}}{-3\left(\frac{1}{5}\right)}\right]_0^{10}$

$= -\frac{1}{2}\left[\frac{1}{\left(\frac{x}{5}-1\right)^3}\right]_0^{10}$

$= -\frac{1}{2}\left[\frac{1}{1} - (-1)\right]$

$= -1.$

(c) $\int_1^k k + x \, dx = 0$

$\left[kx + \frac{x^2}{2}\right]_1^k = 0$

$k^2 + \frac{k^2}{2} - \left(k + \frac{1}{2}\right) = 0.$

$k^2 + \frac{k^2}{2} - k - \frac{1}{2} = 0$

$2k^2 + k^2 - 2k - 1 = 0.$

$3k^2 - 2k - 1 = 0$

$\begin{array}{cc} 3k & +1 \\ k & -1 \end{array}$

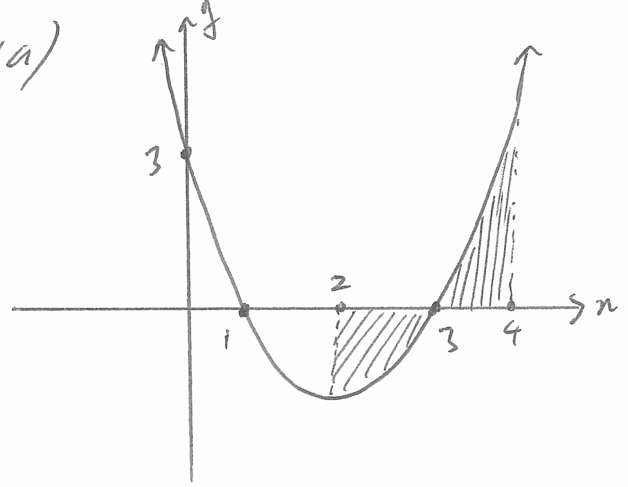
$(3k+1)(k-1) = 0$

$\therefore k = -\frac{1}{3}$ & $k = 1.$

$\therefore k = 1$ only as $k > 0$

Q11 $\quad y = x^2 - 4x + 3 = (x-3)(x-1)$

(a)



$$\int_2^3 x^2 - 4x + 3 \, dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x \right]_2^3$$

$$= (3^2 - 2\times 3^2 + 3^2) - (\tfrac{8}{3} - 8 + 6)$$

$$= -\frac{2}{3}$$

$$\int_3^4 x^2 - 4x + 3 \, dx$$

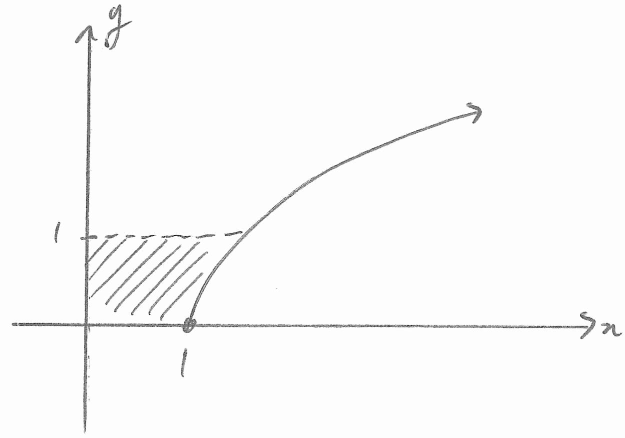
$$= \left[\frac{x^3}{3} - 2x^2 + 3x \right]_3^4$$

$$= \left(\frac{4^3}{3} - 32 + 12\right) - 0$$

$$= \frac{4}{3}$$

$\therefore$ Area $= \frac{2}{3} + \frac{4}{3} = 2$ unit^2

(b)



$$y = \sqrt{x-1}$$

$$y^2 = x - 1$$

$$\therefore x = y^2 + 1$$

$$A = \int_0^1 y^2 + 1 \, dy$$

$$= \left[\frac{y^3}{3} + y \right]_0^1$$

$$= \frac{1}{3} + 1 = \frac{4}{3} \text{ unit}^2$$

(c)

(i) $x = a$ & $x = c$

(ii) $x < a$ & $x > c$

(iii) $b < x < d$.

Q12

(a) $y = ax^3 + bx^2 + cx + d$

$y' = 3ax^2 + 2bx + c$

only one stationary point implies $3ax^2 + 2bx + c = 0$ only has one solution, which implies $\Delta = 0$

$\Delta = (2b)^2 - 4(3a)(c) = 0$

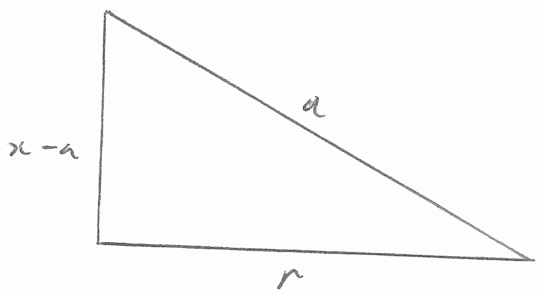
$4b^2 - 12ac = 0$

$b^2 - 3ac = 0$

$\therefore b^2 = 3ac$

(b)

(i)



$r^2 = a^2 - (x-a)^2$

$r^2 = a^2 - (x^2 - 2ax + a^2)$

$r^2 = a^2 - x^2 + 2ax - a^2$

$r^2 = 2ax - x^2$

(ii) $V = \frac{1}{3}\pi r^2 x$

$V = \frac{1}{3}\pi x(2ax - x^2)$

$V = \frac{1}{3}\pi(2ax^2 - x^3)$

(iii)

$\frac{dV}{dx} = \frac{1}{3}\pi(4ax - 3x^2)$

$\frac{d^2V}{dx^2} = \frac{1}{3}\pi(4a - 6x)$

$\frac{dV}{dx} = 0$ when $\frac{1}{3}\pi(4ax - 3x^2) = 0$

$\frac{1}{3}\pi x(4a - 3x) = 0$

$\therefore x = 0$ or $x = \frac{4a}{3}$

$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{4a}{3}} = \frac{1}{3}\pi\left[4a - 6\left(\frac{4a}{3}\right)\right]$

$= \frac{1}{3}\pi(4a - 8a)$

$= -\frac{4a\pi}{3}$

Now $a > 0 \therefore -\frac{4a\pi}{3} < 0$

$\therefore x = \frac{4a}{3}$ gives max V.

Note: $\left.\frac{d^2V}{dx^2}\right|_{x=0} = \frac{4a\pi}{3} > 0 \therefore x = 0$ gives min V

(c)

Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$

$\therefore \int_{-1}^{2} |x| + x^2\,dx = \int_{-1}^{0} -x + x^2\,dx + \int_{0}^{2} x + x^2\,dx$

$= \left[\frac{-x^2}{2} + \frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{0}^{2}$

$= 0 - \left(-\frac{1}{2} - \frac{1}{3}\right) + \left(2 + \frac{8}{3}\right) - 0$

$= \frac{5}{6} + \frac{14}{3} = \frac{11}{2}.$