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Name: ______________________________

Teacher's Name: ______________________________

3unit (1)

MATHEMATICS EXTENSION 1

YEAR 12

ASSESSMENT 1 27 MARCH, 2003

WEIGHTING: **26%**
TIME ALLOWED: **70 MINUTES**

MARKING GUIDELINES:
THE MARKS FOR EACH PART ARE INDICATED BESIDE THE QUESTION

Instructions:

- All questions should be attempted.
- All necessary working must be shown.
- Start each question on a new page
- Put your name and your teacher's name on each page
- Marks may be deducted for careless or untidy work.
- Approved calculators may be used.
- **DO NOT** staple different questions together.
- All rough working paper must be attached to the end of the last question.
- A standard integral sheet is attached.
- **Staple a coloured sheet of paper to the back of each question**
- Hand in this question paper with your answers
- There are four (4) questions in this paper

QUESTION 1.

(a) Find (i) $\frac{d}{dx}\tan^2 x$ **1**

(ii) $\frac{d}{dx}e^{\cos x}$ **1**

(b) Prove the identity

$$\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x} = \sec x$$ **3**

(c) Given that $f'(x) = x + \frac{1}{x}$ and $f(1) = 0$ find $f(e)$. **3**

(d) You are given that the expression $4e^{x} + e^{-x}$ has a minimum value. Find, in simplest log form, the value of x for which this minimum occurs, and find the minimum value, also in simplest form. (You do not need to establish that a minimum occurs.) **4**

QUESTION 2

(a) (i) Find the number and nature of any stationary points of the curve $f(x) = e^x - ex$ **3**

Hence sketch the curve for all real x.

(ii) Using your graph, or otherwise, explain why $e^x \geq ex$ for all real x, and discuss for what value(s) of x $e^x = ex$. **2**

(b) (i) Prove that the equation of the normal PT to the parabola $x^2 = 4ay$ at the point P $(2ap, ap^2)$ is **2**

$$x + py = ap^3 + 2ap$$

(ii) Show that the line QS through the focus S and parallel to the tangent at P is **1**

$$px - y + a = 0$$

(iii) Show that R, the point of intersection of PT and QS is the point **2**

$$\left(ap,\ a(p^2 + 1)\right).$$

(iv) Hence find the equation of the locus of R and discuss briefly the nature of this locus. **2**

QUESTION 3

(a) Solve $\cos\theta = \tan\theta$ for $0 \le \theta \le \pi$. **3**

Give your answer to 2 decimal places.

(b) Consider the equation $\sin x + 3\cos x = 1$ for $-\pi \le x \le \pi$.

(i) Using the formulae for $\sin x$ and $\cos x$ in terms of $t = \tan\frac{x}{2}$, **4**

solve the equation.

(ii) Express $\sin x + 3\cos x$ in the form $R\sin(x+\alpha)$ where **5**

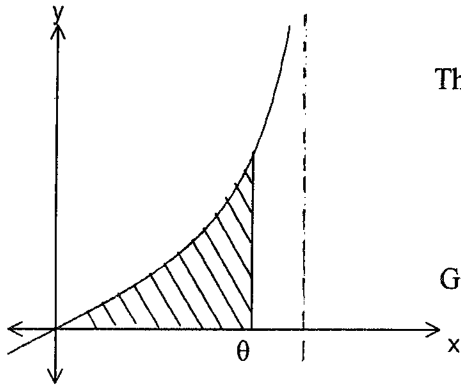
α is acute, and hence sketch $y = \sin x + 3\cos x$ over the given domain. Indicate clearly how your graph could be used to solve the equation in (i) and show your solutions on the graph.

QUESTION 4

(a) Find $\int \frac{4x+1}{2x+1}\,dx$ using the substitution $u = 2x+1$. **3**

(b) Evaluate $\int_0^3 x\sqrt{4-x}\,dx$ using the substitution $u = 4 - x$. **3**

(c) The curve $y = \tan x$ is given for $0 \le x < \frac{\pi}{2}$. **3**



The size of the shaded area is given by

$$\int_0^{\theta} \tan x\,dx = \frac{1}{2}\log 2.$$

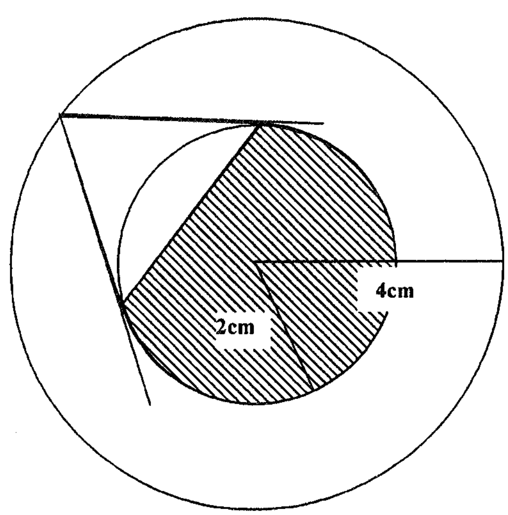
Given that $\tan x = \frac{\sin x}{\cos x}$, find θ.

(d) Two concentric circles have radii 2cm and 4cm. **3**

Two tangents are drawn from a point on the larger circle to the smaller circle, and the points of contact are joined as shown.

Find the exact area of the major segment thus formed, as shaded on the diagram.

[You will need to use: the tangent is perpendicular to the radius at the point of contact.]



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EXT 1 ASS 1 MARCH 2003

Q1 [2]

(a) (i) $\frac{d}{dx}\tan^2 x = 2\tan x\sec^2 x$ (1) R/W

(ii) $\frac{d}{dx}e^{\cos x} = -\sin x\,(e^{\cos x})$ (1) R/W.

(b) $\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x}$ ½ any correct exp of $\cos 2x$.

exp. sin 2x ½

[3]

$LHS = \frac{2\sin x\cos x}{\sin x} - \frac{2\cos^2 x - 1}{\cos x}$ (1)

$= \frac{2\cos^2 x - 2\cos^2 x + 1}{\cos x}$ (1)

$= \frac{1}{\cos x}$ (1)

$= \sec x$ as required.

$= RHS$

(c) $f'(x) = x + \frac{1}{x}$

$f(x) = \frac{1}{2}x^2 + \ln x + (c)$ (1) ½ and ½

but $f(1) = 0$ ∴

$\frac{1}{2}(1)^2 + \ln 1 + c = 0$

[3]

$\frac{1}{2} + 0 + c = 0 \Rightarrow c = -\frac{1}{2}$ (1)

$\therefore f(x) = \frac{1}{2}x^2 + \ln x - \frac{1}{2}$

$f(e) = \frac{1}{2}e^2 + \ln e - \frac{1}{2}$ } ½

$= \frac{1}{2}e^2 + \frac{1}{2}$ } ½ (1)

(d) $\frac{d}{dx}\left(4e^x + e^{-x}\right) = 4e^x - e^{-x}$ (½)

$= 0$ when $e^{-x}(4e^{2x} - 1) = 0$ (½)

[4]

$e^{-x} \neq 0$ (½) $\therefore e^{2x} = \frac{1}{4}$ (½)

$2x = -\ln 4$ (½)

$x = -\ln 2$ (or $\ln\frac{1}{2}$)

when $x = -\ln 2$, $4e^x + e^{-x}$

$= 4e^{-\ln 2} + e^{\ln 2}$ (½)

$= 4\left(\frac{1}{2}\right) + 2$ (1)

$= 4$

2 (a) (i) $f(x) = e^x - ex$

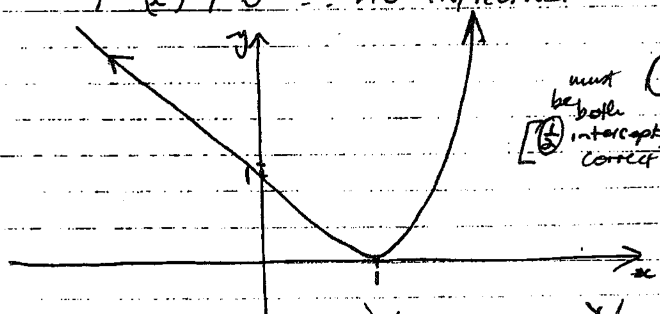
$f'(x) = e^x - e$ ½

$= 0$ when $x = 1$ ½ (1)

$f''(x) = e^x$

$f''(1) > 0$ ∴ $(1, 0)$ is a min T.P. (1) ½ ½

$f''(x) \neq 0$ ∴ no inflection. ignore



must be both (½) intercepts correct, (½) shape cup. (1)

[5]

½✓ ½✓

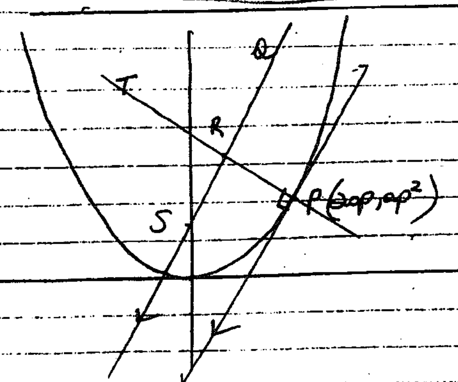
(ii) since $f(x) \geq 0$ for all x, $e^x - ex \geq 0$ (1)

i.e. $e^x \geq ex$

or from the graph. $f(x) = 0$ for one value only

∴ $e^x = ex$ when $x = 1$ only (1)

(b)



so $\text{grad}_{\text{normal}} = -\frac{1}{p}$ (1)

normal eqn: $\frac{y - ap^2}{x - 2ap} = -\frac{1}{p}$

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap$ (1)

(ii) $S \equiv (0, a)$ $\text{grad}_{QS} = p$ ½

∴ eqn is: $\frac{y - a}{x - 0} = p$ ½ (1)

∴ $px = y - a$

$px - y + a = 0$

(iii) $x + py = ap^3 + 2ap$ [1]

$px - y + a = 0$ [2]

From [2] $y = px + a$

In [1] $x + p(px + a) = ap^3 + 2ap$

[7]

$x(1 + p^2) + ap = ap^3 + 2ap$ (1)

$x = \frac{ap^3 + ap}{1 + p^2}$

$= \frac{ap(p^2 + 1)}{1 + p^2}$ (½)

$= ap$

∴ $y = p(ap) + a$

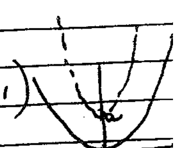
$= ap^2 + a$ (½)

$= a(p^2 + 1)$

so $R \equiv (ap, a(p^2 + 1))$

(iv) $x = ap \Rightarrow p = \frac{x}{a}$ so $y = a\left(\left(\frac{x}{a}\right)^2 + 1\right)$

(1) $y = \frac{x^2 + a^2}{a}$



or drawn

This is a parabola vertex $(0, a)$ inside

3 (a) $\cos\theta = \tan\theta \qquad 0 \le \theta \le \pi$

$\cos\theta = \frac{\sin\theta}{\cos\theta}$ ½

$\cos^2\theta - \sin\theta = 0$

$1 - \sin^2\theta - \sin\theta = 0$ ½

$\sin^2\theta + \sin\theta - 1 = 0$

$\sin\theta = \frac{1 \pm \sqrt{1-4(-1)}}{2}$ ½ (1)

[3] $= \frac{-1+\sqrt5}{2}$ or $\left(\frac{-1-\sqrt5}{2}\text{ — rejected}\right)$

$\theta = 0.6662...$ or $\pi - 0.6662...$

≈ 0.67 or $2.475...$

$\approx 0.67^c$ or 2.48^c (1)

38.1° 141.83°

(b)(i) $\sin x + 3\cos x = 1 \qquad -\pi \le x \le \pi$

$\frac{2t}{1+t^2} + 3\cdot\frac{1-t^2}{1+t^2} = 1$ where $t = \tan\frac{x}{2}$

$2t + 3 - 3t^2 = 1 + t^2$

$4t^2 - 2t - 2 = 0$

$2t^2 - t - 1 = 0$

$(2t+1)(t-1) = 0$

[4] $t = -\frac12$ or 1 (1) sign for x:

$\tan\frac{x}{2} = \frac{-1}{2}$ or $\tan\frac{x}{2} = 1$

$\frac{x}{2} = -0.4636...$ (1) $\frac{x}{2} = \frac{\pi}{4}$ (1)

$-.296\pi$

$x = -0.927$ or $x = \frac{\pi}{2}$

$= -53.13°$ $= 90°$

(ii) $\sin x + 3\cos x \equiv R\sin(x+\alpha)$

$= R(\sin x\cos\alpha + \cos x\sin\alpha)$

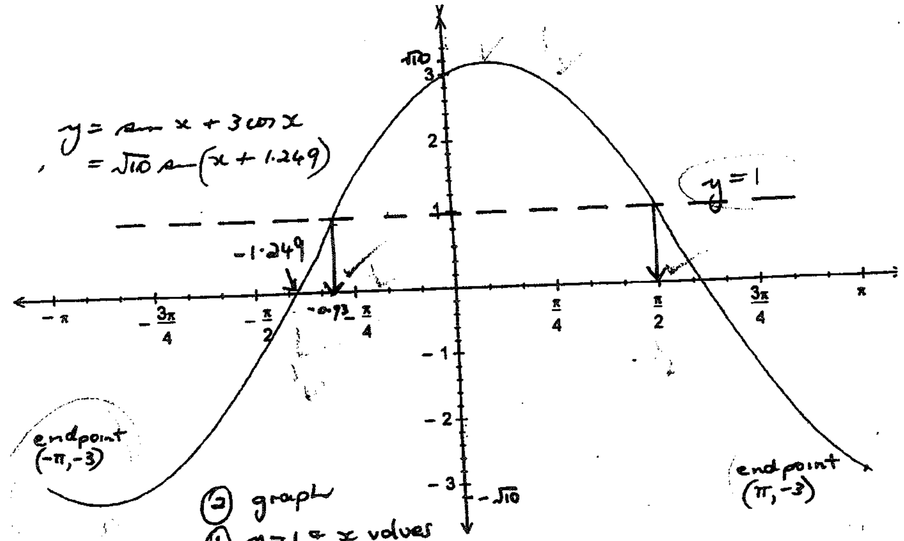
$\therefore R\cos\alpha = 1$, $R\sin\alpha = 3$ } Square + add: $R = \sqrt{10}$ ½ (1)

don't accept $\tan^{-1}3$. deduct ½.

.398π

[5]

Hence



3U ASS 1
2003 RC

Q6 (a) $\int \frac{4x+1}{2x+1}\,dx$ $\qquad u = 2x+1$, $\; 2u-1 = 4x+1$, $\; du = 2\,dx$

$= \int \frac{2u-1}{u}\cdot\frac{du}{2}$ (1)

[3] $= \frac{1}{2}\int \left(2 - \frac{1}{u}\right) du$ (1)

$= \frac{1}{2}\left(2u - \ln u\right) + C$ (1)

$= 2x+1 - \frac{1}{2}\ln(2x+1) + C$

(b) $\int_0^3 x\sqrt{4-x}\,dx$ $\qquad u = 4-x$, $\; du = -dx$, $\; x=0,\ u=4$; $\; x=3,\ u=1$

$= \int_4^1 (4-u)\sqrt{u}\,.\,-du$ (1)

$= \int_1^4 \left(4u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du$

[3] $= \left[\frac{2.4u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{5}{2}}}{5}\right]_1^4$ (1)

$= \left(\frac{8}{3}\times 8 - \frac{64}{5}\right) - \left(\frac{8}{3} - \frac{2}{5}\right)$ (1)

$= \frac{94}{15}$

(c) $\int_0^\theta \tan x\,dx = -\ln\cos x\Big]_0^\theta$ (1)

$= -\ln\cos\theta + \ln\cos 0 = \frac{1}{2}\ln 2$

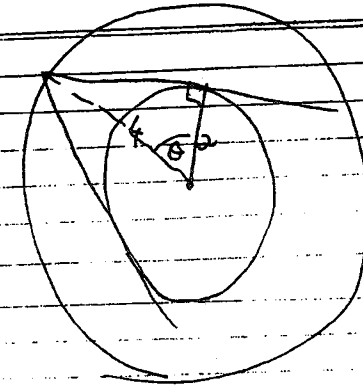
[3] so $\ln\frac{1}{\cos\theta} + \ln 1 = \frac{1}{2}\ln 2$

$\ln\frac{1}{\cos\theta} = \ln\sqrt{2}$ (1)

$\therefore \cos\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$ (1) $\left(0 \le \theta < \frac{\pi}{2}\right)$

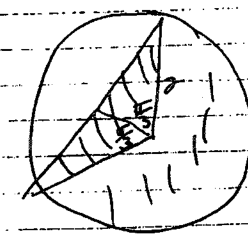
(d)



$\cos\theta = \frac{2}{4}$

$\theta = \frac{\pi}{3}^c$ (1)

[3]



$A_\Delta = \frac{1}{2}(2)(2)\sin\frac{2\pi}{3}$

$= 2\times\frac{\sqrt{3}}{2}$

$= \sqrt{3}$ (1)

$A_{sector} = \frac{1}{2}(2)^2\cdot\frac{4\pi}{3}$

$= \frac{8\pi}{3}$ (1)

$\therefore$ Total shaded area $= \sqrt{3} + \frac{8\pi}{3}$ sq. units