THE SCOTS COLLEGE

YEAR 12

EXTENSION 2 MATHEMATICS

ASSESSMENT TASK 1

FEBRUARY 2004

WEIGHTING: **10%**

TIME ALLOWED: **50 MINUTES**

INSTRUCTIONS:

- ALL NECESSARY WORKING MUST BE SHOWN.
- BOARD APPROVED CALCULATORS MAY BE USED.
- START EACH QUESTION ON A NEW PAGE.

OUTCOMES:

E3 : Uses the relationship between algebraic and geometric representations of complex numbers and of conics.

E6 : Combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions.

QUESTION 1 **[MARKS]**

(a) **(i)** Express $-1+\sqrt{3}i$ in modulus - argument form. **[1]**

(ii) Hence evaluate $\left(-1+\sqrt{3}i\right)^9$ **[2]**

(b) **(i)** Find the value of the product $\left(-1+\sqrt{3}i\right)(1+i)$ **[1]**

(ii) Hence, or otherwise, find the exact value of $\cos\frac{11\pi}{12}$. **[2]**

(c) Sketch on separate Argand diagrams, the locus of z defined by:

(i) $\arg\left(\frac{z-i}{z+1}\right)=0$ **[2]**

(ii) $\left|z-(2+3i)\right|=5$ **[2]**

(d) If $z_1=a+ib$ and $z_2=c+id$

(i) Show algebraically that $|z_1z_2|=|z_1||z_2|$ and from the Argand diagram explain why $|z_1+z_2|\le|z_1|+|z_2|$. **[4]**

(ii) Hence or otherwise, if $|z|\le\frac{1}{2}$, show that $\left|(1+i)z^3+iz\right|<\frac{3}{4}$. **[3]**

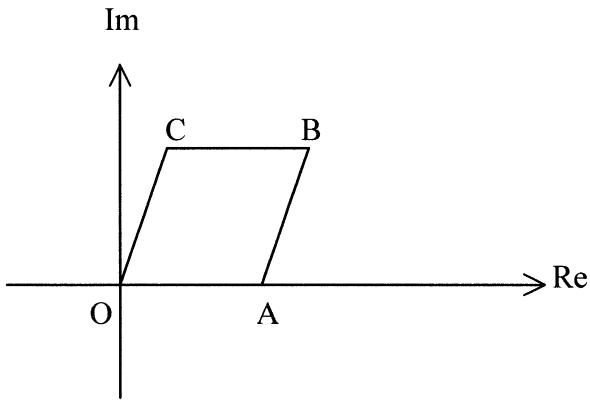
(e) $1-2i$ is the root of the equation $z^2-(3+i)z+c=0$.

(i) Explain why the conjugate $1+2i$ cannot be a root to the equation. [1]

(ii) Show that the other root is $2+3i$. [1]

(iii) Find the value of c. [1]

(f) OABC is a rhombus.
O lies at the origin.
A is on the real axis.
C corresponds to the complex number $1+\sqrt{3}i$.



(i) Find the complex number corresponding to the point B. [1]

(ii) If the figure is rotated anti-clockwise by $\frac{\pi}{3}$ radians about O to form a new rhombus $OA'B'C'$, draw this new rhombus on an Argand diagram and find the complex number corresponding to B'. [2]

(a) Let $f(x)=(x-1)(x-3)^2$. Sketch each of the following on separate diagrams:

(i) $y=f(x)$ [1]

(ii) $|y|=f(x)$ [3]

(iii) $y=f(|x|)$ [2]

(iv) $y^2=f(x)$ [3]

(b) Consider the curve $f(x)=4x^2(2-x^2)$

(i) Find the stationary points of $y=f(x)$ and determine their nature. [3]

(ii) Sketch the graph of $y=f(x)$, showing the x intercepts. [2]

(iii) Also sketch $y=ln\,f(x)$ [3]

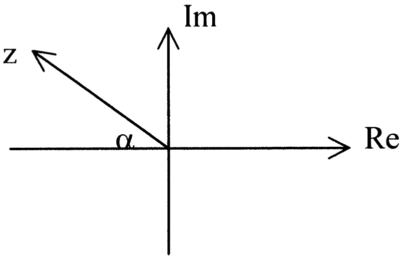
Ext 2 Maths.

SOLUTIONS - FEBRUARY 2004

QUESTION 1

[MARKS]

(a) **(i)** $2 = -1+\sqrt{3}i$



$|z| = \sqrt{1+3}$

$= 2$

½ for mod

$\tan\alpha = \dfrac{\sqrt{3}}{1}$

$\alpha = \dfrac{\pi}{3}$

$\arg(z) = \pi - \dfrac{\pi}{3}$

$= \dfrac{2\pi}{3}$

½ for arg

$\therefore z = 2cis\dfrac{2\pi}{3}$

(ii) $\left(-1+\sqrt{3}i\right)^9 = 2^9\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)^9$

$= 2^9\left(\cos 6\pi + i\sin 6\pi\right)$

$= 2^9 = 512$

1 De Moivre's
1 answer

(b) **(i)** $\left(-1+\sqrt{3}i\right)(1+i)$

$= -1 - i + \sqrt{\ }\, i - \sqrt{3}$

$= \left(-1-\sqrt{3}\right) + \left(\sqrt{3}-1\right)i$

½
½ answer

(ii) $\left(-1+\sqrt{3}i\right)(1+i)$

$= 2\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)\sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$ ½

$= 2\sqrt{2}\left(\cos\left(\dfrac{2\pi}{3}+\dfrac{\pi}{4}\right) + i\sin\left(\dfrac{2\pi}{3}+\dfrac{\pi}{4}\right)\right)$

$= 2\sqrt{2}\left(\cos\dfrac{11\pi}{12} + i\sin\dfrac{11\pi}{12}\right)$ ½

equating real parts

$2\sqrt{2}\cos\dfrac{11\pi}{12} = -1-\sqrt{3}$ ½

$\therefore \cos\dfrac{11\pi}{12} = \dfrac{-1-\sqrt{3}}{2\sqrt{2}}$ ½

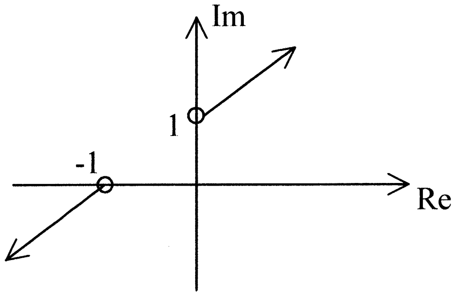
$= \dfrac{-\sqrt{2}-\sqrt{6}}{4}$

[MARKS]

(c) **(i)** $\arg\left(\dfrac{z-i}{z+1}\right) = 0$ ½

$\arg(z-i) - \arg(z+1) = 0$ ½

i.e. $\arg(z-i) = \arg(z+1)$



1

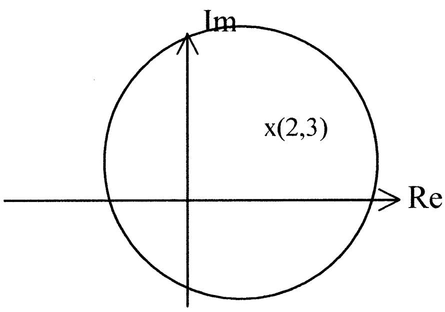
$\therefore$ line $y = x + 1$ but not in the interval $-1 < x < 0$.

(ii) $\left|z - (2+3i)\right| = 25$

$\left|(x-2) + (y-3)i\right| = 25$ ½

$(x-2)^2 + (y-3)^2 = 25$ ½

$\therefore$ circle centre (2, 3) radius 5 units ½



½

(d) **(i)** $z_1 = a + ib, z = c + id$

LHS $= |z_1, z_2|$

$= |(a+ib)(c+2d)|$ 1

$= |ac + adi + bci - bd|$

$= |(ac - bd) + i(d + bc)|$

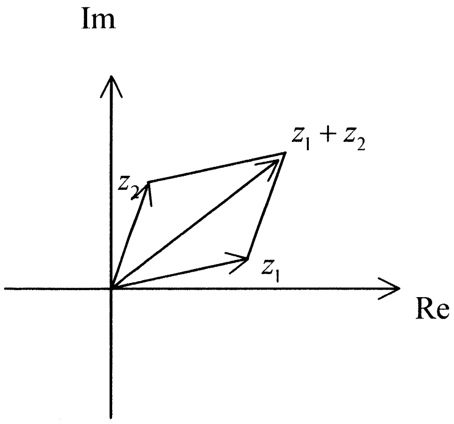
$= \sqrt{a^2c^2 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2}$ 1

$= \sqrt{a^2(c^2+d^2) + b^2(c^2+d^2)}$

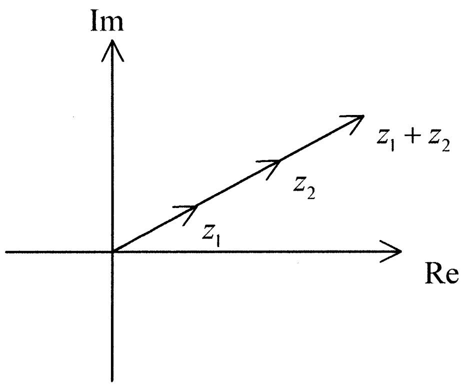
$= \sqrt{(a^2+b^2)(c^2+d^2)}$ 1

$= \sqrt{a^2b^2} \,.\, \sqrt{c^2+d^2}$

$= |z_1| \,.\, |z_2|$



The longest side of a Δ is less than the sum of the other 2 sides. ½
i.e. $|z_1 + z_2| < |z_1| + |z_2|$



However, when the $\arg(z_1) = \arg(z_2)$ then
$|z_1 + z_2| = |z_1| + |z_2|$ ½
$|z_1 + z_2| \le |z_1| + |z_2|$

(ii) $\left|(1+i)z^3 + iz\right|$ ½

$= |z|\left|z^2(1+i)+i\right|$

$\le \frac{1}{2}\left[\left|z^2\right|\left|1+i\right| + |i|\right]$ ½

$\le \frac{1}{2}\left[\frac{1}{4}\left(\sqrt{2}+1\right)\right]$ ½

$\le \frac{1}{2}\left(\frac{\sqrt{2}+4}{4}\right)$ ½

$= \frac{\sqrt{2}+4}{8}$

$< \frac{6}{8}$ since $\sqrt{2} = 1.42$ …… ½

$< \frac{3}{4}$ ½

(e) $z^2 - (3+i)z + C = 0$

(i) Conjugate $1+2i$ cannot be a root since the co-efficient are not real. 1

(ii) Let roots be $\alpha, (1-2i)$

sum of roots $= \frac{-b}{a}$ ½

$\alpha + 1 - 2i = 3 + i$ ½

$\alpha = 2 + 3i$

[MARKS]

(iii) Product of roots $= \dfrac{c}{a}$

$(1-2i)(2+3i) = c$ ½

$2+3i-4i+6 = c$ ½

$\therefore c = 8-i$

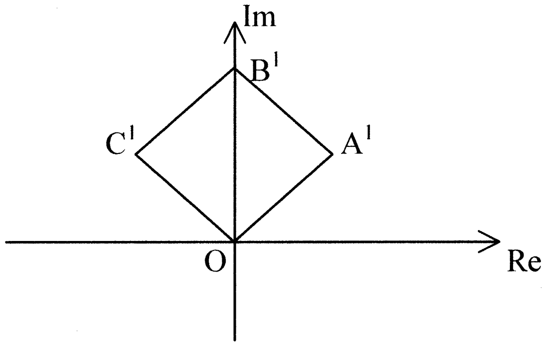
(f) **(i)** Let B be $z = x+iy$

$|z| = \sqrt{1^2 + \sqrt{3}^2}$ $\qquad \arg(z) = \dfrac{\pi}{3}$ ½

$= 2$

$\therefore$ B is point $1+\sqrt{3}i+2 = 3+\sqrt{3}i$ ½

(ii)



1 diagram

OB $= \sqrt{3^2 + (\sqrt{3})^2}$ ½

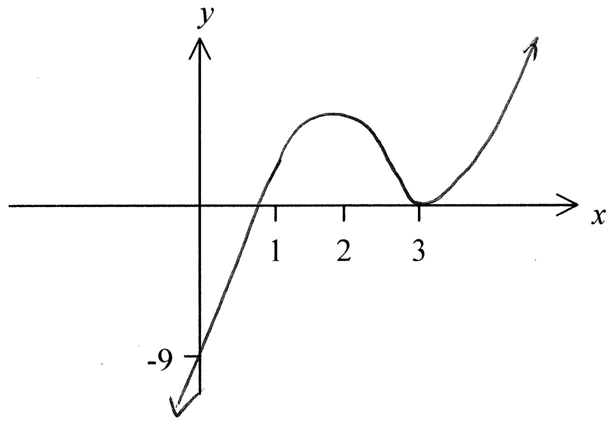
$= \sqrt{12}$

$= 2\sqrt{3}$

$\therefore$ B' is $2\sqrt{3}i$ ½

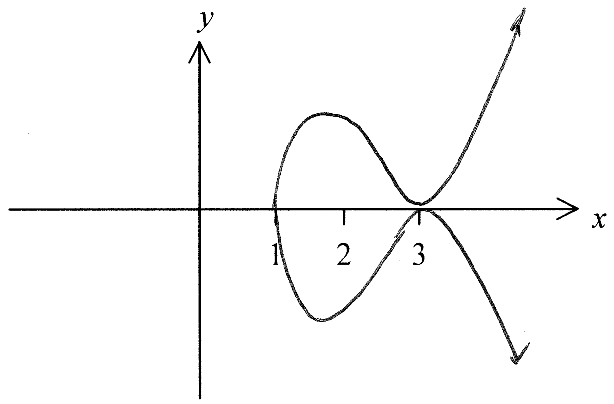
QUESTION 2 **[MARKS]**

(a) **(i)** $y = f(x) = (x-1)(x-3)^2$



1
diagram

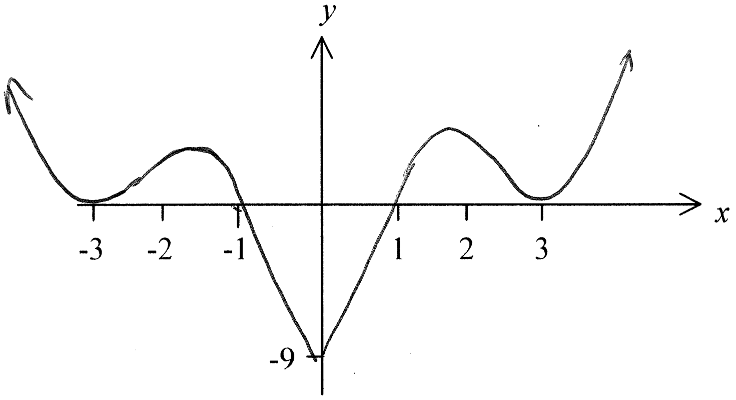
(ii) $|y| = f(x)$



3
correct
diagram

2
half graph

(iii) $y = f(|x|)$



2
correct
diagram

1
variations

[MARKS]

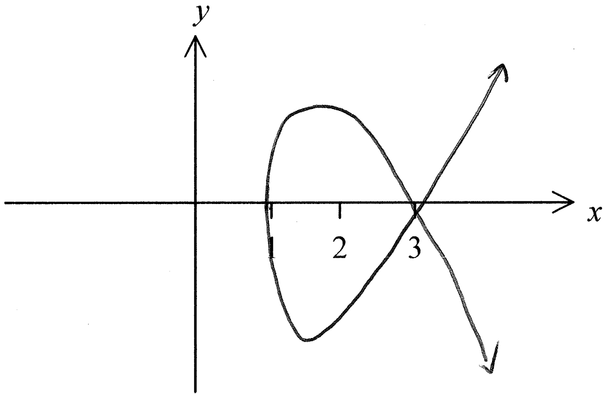
(iv) $y^2 = f(x)$

$$2y\frac{dy}{dx} = f^1(x)$$

$$\frac{dy}{dx} = \frac{f^1(x)}{2y}$$

$\therefore$ Stat pts at $f^1(x) = 0,\ x = 2$

$\therefore$ critical pts, $y = 0$, $x = 1, 3$



3
correct
diagram

[MARKS]

(b) $f(x) = 4x^2(2 - x^2) = 8x^2 - 4x^4$

(i) $f'(x) = 16x - 16x^3$ — **1** 1st +2nd derivatives

$f''(x) = 16 - 48x^2$

For stationary points, $f'(x) = 0$

$16x(1 - x^2) = 0$ — **1** find stat points

$x = 0, \ x = 1, \ x = -1$

At $x = -1, \ f(x) = 4, \ f''(x) < 0 \cap$

$(-1, 4)$ max turning point.

At $x = 0, \ f(x) = 0, \ f''(x) > 0 \cup$

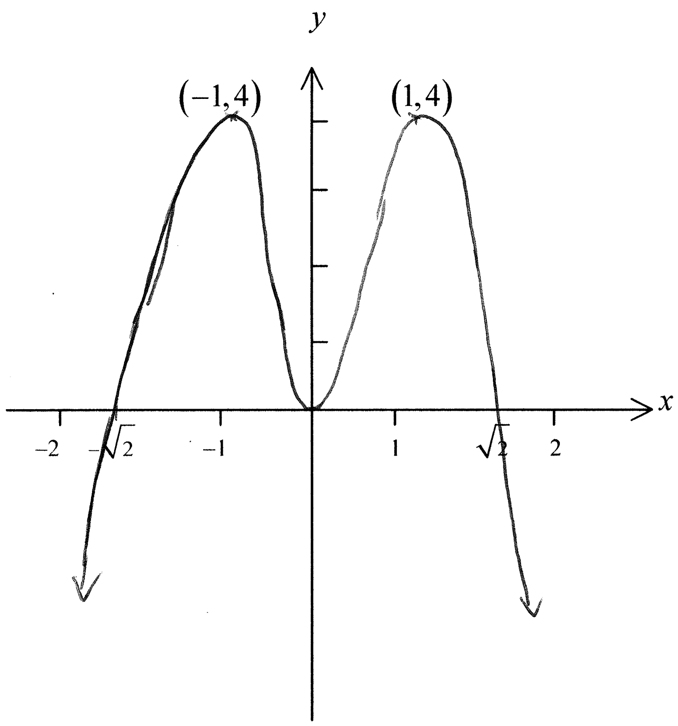
$(0, 0)$ min turning point. — **1** determining their nature

At $x = 1, \ f(x) = 4, \ f''(x) < 0 \cap$

$(1, 4)$ max turning point.

(ii) Sub $y = 0$, $4x^2(2 - x^2) = 0$

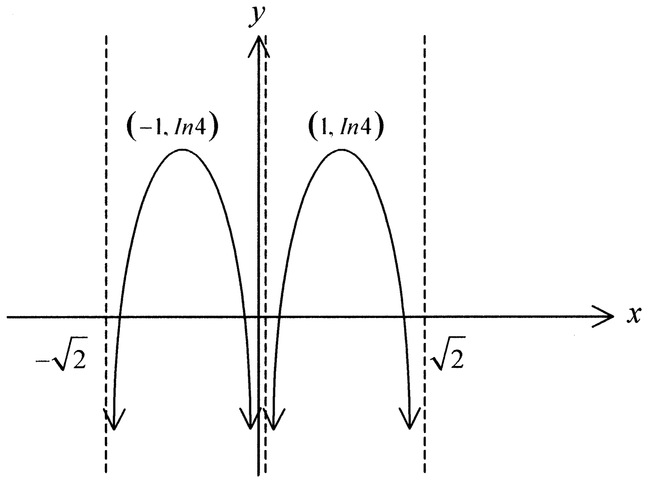
$x = 0, \quad x = \pm\sqrt{2}$ — **2** complete graph + intercepts



[MARKS]

(iii) $y = In\,f(x)$

Domain $-\sqrt{2} < x < 0$ and $0 < x < \sqrt{2}$



1 asymptotes

1 shape of graph

1 scale

$$\frac{dy}{dx} = \frac{f^{1}(x)}{f(x)}$$

Stat points,	$f^{1}(x) = 0$,	i.e. ~~$x = 0$~~,	$x = \pm 1$
Critical points	$f(x = 0)$,	i.e. $x = 0$,	$x = \pm\sqrt{2}$