GOSFORD HIGH SCHOOL

2009 HIGHER SCHOOL CERTIFICATE

MATHEMATICS

ASSESSMENT TASK 3

Time Allowed – 70 minutes + 5 minutes reading time

All necessary working should be shown.

Full marks may not be awarded for unnecessarily untidy work or work that is poorly organized.

Students must begin each new question on a new page.

Questions will be collected separately at the conclusion of the assessment task.

All questions are to be attempted.

QUESTION 1 (18 marks) START A NEW PAGE

a) The first 4 terms of a sequence are 204, 198, 192, 186

(i) Find the 50^{th} term of the sequence. (2)

(ii) Find the sum of the first 50 terms (2)

b) The first term of an infinite geometric sequence is 8 and the sum to infinity is 32.

Find the common ratio. (2)

c) Consider the sequence $\{\sqrt{2}+1,\ \sqrt{8}-1,\ \sqrt{18}-3,\ \ldots\ldots\ldots\ldots\}$

Show that the sequence is an Arithmetic sequence. (2)

d) For a particular series $S_n = 3n^2 - 2n$.

i) Find S_{25} (1)

ii) Find an expression for T_n in simplest form. (2)

iii) Find the first value of n for which $S_n > 1000$. (3)

e) (i) Write down the formula for finding the sum to n terms (S_n) of a geometric series with first term (a) and common ratio(r) where $|r| > 1$ (1)

(ii) Find $\sum_{k=1}^{20}\left[\frac{2^k}{5}\right]$ (3)

QUESTION 2 (13 marks) START A NEW PAGE

a) x , 2 , y form an ~~Arithmetic~~ Geometric sequence, whilst 2 , y , $28x$ form a ~~Geometric~~ Arithmetic sequence.

Find the possible values of x and y. (4)

b) Alec borrows \$50000 and undertakes to repay \$R at the end of each month, calculated from the date of the loan. Interest is charged on the unpaid debt at the rate of 6% p.a. compounded monthly.

The amount owing at the end of the first month A_1 is given by $A_1 = \$\{50000(1 \cdot 005) - R\}$

(i) Prove that the amount owing at the end of the 2nd month A_2 is given by

$$A_2 = \$\{50000(1 \cdot 005)^2 - R[1 + (1 \cdot 005)]\}$$ (2)

(ii) Derive a similar expression for A_3, the amount owing at the end of the 3rd month (2)

(iii) Prove that, A_n the amount owing at the end of n months is given by

$$A_n = 50000(1 \cdot 005)^n - \frac{R(1 \cdot 005^n - 1)}{0 \cdot 005}$$ (2)

(iv) Calculate n, correct to the nearest whole number, given $R = 1200$ and $A_n = 0$. (3)

QUESTION 3 (15 marks) START A NEW PAGE

a) Using the table of standard integrals, find $\int \sec 2x \tan 2x \, dx$ (1)

b) Find $\int \sin(2x+3)\, dx$ (2)

c) Find $\frac{d}{dx}\left[e^{\tan x}\right]$ (1)

d) Differentiate $\sin^2 x$ (1)

e) Find the equation of the tangent to the curve $y = x\cos x$ at the point $(\frac{\pi}{2}, 0)$ (3)

f) Consider the function $f(x) = 3\sin(2\pi x)$ where $0 \le x \le 1$.

(i) State the range of the function. (1)

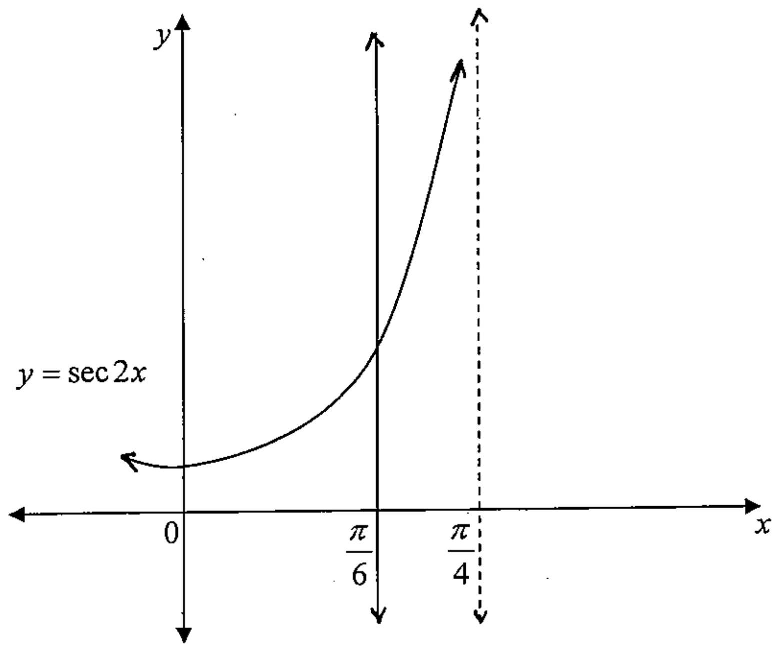
(ii) State the period of the curve $y = f(x)$. (1)

(iii) Sketch the curve $y = f(x)$. (2)

g) The diagram shows part of the graph of the function $y = \sec 2x$.

The region bounded by the curve, the x axis and the line $x = \frac{\pi}{6}$ is rotated about the x axis to form a solid.
and the y axis

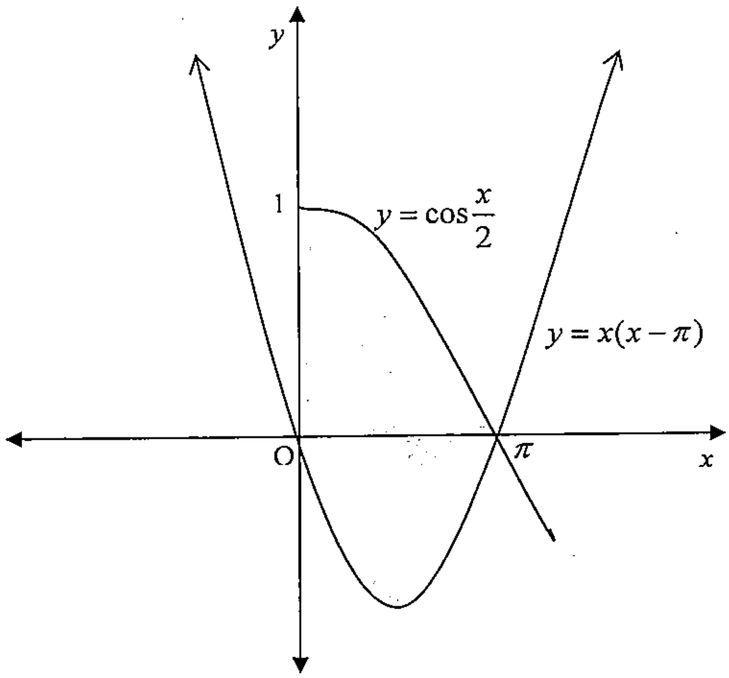
Find the exact volume of the solid. (3)



QUESTION 4 (14 marks) START A NEW PAGE

a) Solve $2\cos^2 x + \cos x = 0$, for $0 \le x \le 2\pi$ (3)

b) The shaded area is bounded by the y axis and the curves $y = x(x-\pi)$ and $y = \cos\frac{x}{2}$ for $0 \le x \le \pi$.
Find the exact value of this area. (4)



c) (i) Find $\frac{d}{dx}[1-\cos 2x]$ (1)

(ii) Hence, or otherwise, find $\int \frac{\sin 2x}{1-\cos 2x}\,dx$ (2)

d) Consider the function $y = 1 + \sin x + \sqrt{3}\cos x$

Find the minimum value $1 + \sin x + \sqrt{3}\cos x$ in the interval $0 \le x \le 2\pi$ (4)

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

2009 HSC MATHEMATICS ASSESSMENT TASK 3 SOLUTIONS

Question 1

a) Sequence is Arithmetic with $a = 204$, $d = -6$

$T_n = a + (n-1)d$

$T_{50} = 204 + 49 \times (-6)$ (1)

$= -90$ (1)

$S_n = \frac{n}{2}[a + l]$

$S_{50} = \frac{50}{2}[204 + (-90)]$ (1)

$= 2850$ (1)

b) $S = \frac{a}{1-r}$

$32 = \frac{8}{1-r}$ (1)

$1 - r = \frac{8}{32}$

$1 - \frac{1}{4} = r$

$r = \frac{3}{4}$ (1)

c) $T_2 - T_1 = \sqrt{8} - 1 - (\sqrt{2} + 1)$

$= 2\sqrt{2} - 1 - \sqrt{2} - 1$

$= \sqrt{2} - 2$

$T_3 - T_2 = \sqrt{18} - 3 - (\sqrt{8} - 1)$

$= 3\sqrt{2} - 3 - 2\sqrt{2} + 1$

$= \sqrt{2} - 2$

Since $T_2 - T_1 = T_3 - T_2$ (2)

$\therefore$ Sequence is Arithmetic

d) $S_n = 3n^2 - 2n$

i) $S_{25} = 3(25)^2 - 2(25)$

$= 1825$ (1)

ii) $T_n = S_n - S_{n-1}$

$= 3n^2 - 2n - [3(n-1)^2 - 2(n-1)]$ (1)

$= 3n^2 - 2n - 3(n^2 - 2n + 1) + 2n - 2$

$= 3n^2 - 3n^2 + 6n - 3 - 2$

$\therefore T_n = 6n - 5$ (1)

iii) If $S_n > 1000$

$3n^2 - 2n > 1000$

$3n^2 - 2n - 1000 > 0$ (1)

Solve $3n^2 - 2n - 1000 = 0$

$n = \frac{2 \pm \sqrt{4 - 4(3)(-1000)}}{6}$ (1)

$n = \frac{2 \pm \sqrt{12004}}{6}$

$n \doteqdot 18 \cdot 59 \longrightarrow n = 19$ (1)

e) i) $S_n = \frac{a(r^n - 1)}{r - 1}$ can accept $S_n = \frac{a(1 - r^n)}{1 - r}$ (1)

ii) $\sum_{k=1}^{20} \frac{2}{5}^k = \frac{2}{5}^1 + \frac{2}{5}^2 + \frac{2}{5}^3 + \dots + \frac{2}{5}^{20}$ (1)

$= \frac{0.4(2^{20} - 1)}{1}$ Using $S_n = a(1 - r^n)$ (1)

$= 419430$ (1)

Question 2

$x, 2, y$ is Geometric $\rightarrow \therefore \frac{2}{x} = \frac{y}{2}$

$\therefore xy = 4 \dots (A)$ (1)

$2, y, 28x$ is Arithmetic $\rightarrow y - 2 = 28x - y$

$\therefore 2y = 28x + 2$

$y = 14x + 1 \dots (B)$ (1)

Solving (A) & (B)

$\therefore x(14x + 1) = 4$

$14x^2 + x = 4$

$14x^2 + x - 4 = 0$

$(7x+4)(2x-1)=0$

$\therefore x = -\frac{4}{7}, \frac{1}{2}$

$\therefore y = -7$ and 8 respectively (2)

?) (i) $A_1 = 50000(1.005) - R$

$A_2 = A_1(1.005) - R$

$= \{50000(1.005) - R\}1.005 - R$ (1)

$= 50000(1.005)^2 - R(1.005) - R$ $(*)$

$A_2 = 50000(1.005)^2 - R[1 + (1.005)]$ as required (1)

(ii) $A_3 = A_2(1.005) - R$

$= \{50000(1.005)^2 - R(1.005) - R\}1.005 - R$ using $*$ (1)

$= 50000(1.005)^3 - R(1.005)^2 - R(1.005) - R$

$= 50000(1.005)^3 - R[1 + 1.005 + 1.005^2]$ (1)

(iii) $A_n = 50000(1.005)^n - R[1 + 1.005 + 1.005^2 + \cdots + 1.005^{n-1}]$ (1)

$= 50000(1.005)^n - \frac{R \cdot 1[(1.005)^n - 1]}{1.005 - 1}$ using $S_n = \frac{a(r^n-1)}{r-1}$

$= 50000(1.005)^n - \frac{R(1.005^n - 1)}{0.005}$ (1)

iv) $0 = 50000(1.005)^n - \frac{1200(1.005^n - 1)}{0.005}$

$0 = 250(1.005)^n - 1200(1.005)^n + 1200$

$950(1.005)^n = 1200$

$(1.005)^n = \frac{24}{19}$ (2)

$n = \log_{1.005}\left(\frac{24}{19}\right)$

$n = \frac{\ln\left(\frac{24}{19}\right)}{\ln 1.005}$

$n = 46.83 \rightarrow n = 47$ (1)

Question 3

a) $\int \sec 2x \tan 2x \, dx = \frac{1}{2}\sec 2x + c$ (1)

b) $\int \sin(2x+3)\, dx = -\frac{1}{2}\cos(2x+3) + c$ (2)

c) $\frac{d}{dx}\left[e^{\tan x}\right] = \sec^2 x \cdot e^{\tan x}$ (1)

d) Let $y = (\sin x)^2$

$\frac{dy}{dx} = 2(\sin x)\cos x$

$= 2\sin x \cos x$ (1)

e) $y = x\cos x$

$\frac{dy}{dx} = \cos x \cdot 1 + x \cdot (-\sin x)$

$= \cos x - x\sin x$ (1)

$= 0 - \frac{\pi}{2}$ at $x = \frac{\pi}{2}$

$\frac{dy}{dx} = -\frac{\pi}{2}$ (1)

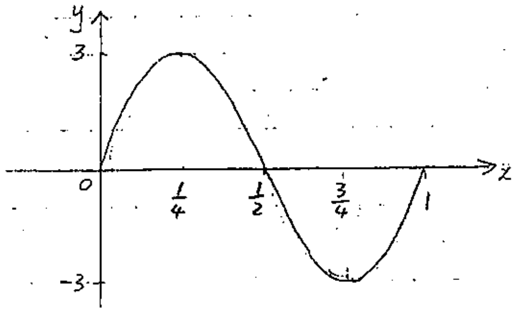
Equation of Tangent is $y - 0 = -\frac{\pi}{2}\left(x - \frac{\pi}{2}\right)$ (1)

$$y = -\frac{\pi x}{2} + \frac{\pi^2}{4}$$

f) i) Range: $-3 \le f(x) \le 3$ (1)

ii) Period $\frac{2\pi}{2\pi} = 1$ (1)

iii)



(2)

g) $V = \pi \int_0^{\frac{\pi}{6}} (\sec 2x)^2 \, dx$

$= \pi \int_0^{\frac{\pi}{6}} \sec^2 2x \, dx$ (1)

$= \pi \left[\frac{1}{2} \tan 2x \right]_0^{\frac{\pi}{6}}$ (1)

$= \frac{\pi}{2} \left[\tan \frac{\pi}{3} - \tan 0 \right]$

$= \frac{\pi\sqrt{3}}{2}$ cubic units (1)

Question 4. a) $\cos x (2\cos x + 1) = 0$

$\therefore \cos x = 0$ or $\cos x = -\frac{1}{2}$ (2)

$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$ (1)

b) Area $= \int_0^{\pi} \cos \frac{x}{2} - x(x - \pi) \, dx$ (1)

$= \int_0^{\pi} \cos \frac{x}{2} - x^2 + \pi x \, dx$

$= \left[2\sin \frac{x}{2} - \frac{x^3}{3} + \frac{\pi x^2}{2} \right]_0^{\pi}$ (2)

$= \left[2 - \frac{\pi^3}{3} + \frac{\pi^3}{2} - 0 \right]$

$= \left[2 - \frac{2\pi^3 - 3\pi^3}{6} \right.$

$= \left[2 + \frac{\pi^3}{6} \right]$ square units (1)

c) (i) $\frac{d}{dx} \left[1 - \cos 2x \right] = 2\sin 2x$ (1)

(ii) $\int \frac{\sin 2x}{1 - \cos 2x} \, dx = \frac{1}{2} \int \frac{2\sin 2x}{1 - \cos 2x} \, dx$

$= \frac{1}{2} \ln(1 - \cos 2x) + c$ (2)

d) $y = 1 + \sin x + \sqrt{3}\cos x$

$\frac{dy}{dx} = \cos x - \sqrt{3}\sin x$ (1) $\frac{d^2y}{dx^2} = -\sin x - \sqrt{3}\cos x$

For Stationary Points $\frac{dy}{dx} = 0$

$\therefore \cos x - \sqrt{3}\sin x = 0$

$\sqrt{3}\sin x = \cos x.$

$\tan x = \frac{1}{\sqrt{3}}$

$x = \frac{\pi}{6}, \frac{7\pi}{6}$ (1)

when $x = \frac{\pi}{6} \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{2} - \frac{3}{2}$

$= -2 < 0$

$\therefore$ Max. Turning Pt. at $x = \frac{\pi}{6}$

when $x = \frac{7\pi}{6} \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2} + \frac{3}{2}$

$= 2 > 0$

$\therefore$ Minimum Turning Pt at $x = \frac{7\pi}{6}$ (1)

$\therefore$ Minimum Value is $1 + \sin\left(\frac{7\pi}{6}\right) + \sqrt{3}\cos\left(\frac{7\pi}{6}\right)$

$= 1 + \left(-\frac{1}{2}\right) + \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right)$

$= 1 - \frac{1}{2} - \frac{3}{2}$

$= -1$ (1)