STUDENT NAME ……………………………………….

YEAR 12 Mathematics
HSC Course
Assessment Task 3
2010

Time Allowed – 70 Minutes

1. There are 4 questions, NOT of equal value.
2. Answer each question on your own paper.
3. Start each Question on a new sheet of paper.
4. Show all necessary working
5. Use one side of the paper only.
6. Calculators may be used

Topic	Mark
1. Question 1 (Trigonometric Functions)	/16
2. Question 2 (Logs and Exponentials)	/17
3. Question 3 (Trigonometric Functions)	/18
4. Question 4 (Logs and Exponentials)	/16
Total	/67

Question 3 (18 Marks) (Start a New Sheet of Paper) **Marks**

a) Find the equation of the normal to the curve $y = 5\sin 2x$ at the point where $x = \frac{\pi}{6}$ **3**
Do not simplify your answer.

b) Find $\lim_{x \to 0} \frac{\sin 3x}{x}$ **2**

c) For $y = 4\cos 3x - 2\sin 3x$, show that $\frac{d^2y}{dx^2} = -9y$ **2**

d) (i) Show that $x = \frac{\pi}{3}$ is a solution to the equation $\sin 2x = \sin x$ **1**

(ii) Sketch on the same axes, the curves $y = \sin 2x$ and $y = \sin x$ for $0 \le x \le 2\pi$. **2**

(iii) Write down 2 more solutions to the equation $\sin 2x = \sin x$ in this domain. **2**

(iv) Find the area between $y = \sin 2x$ and $y = \sin x$ for $\frac{\pi}{3} \le x \le \pi$ **3**

e) Use Simpson's Rule with 5 function values to approximate $\int_0^{\pi} \sin^2 x \, dx$. **3**

Question 4 (16 Marks) Start a New Sheet of Paper **Marks**

a) (i) Differentiate xe^{2x} **2**

(ii) Hence find $\int xe^{2x} dx$ **3**

c) For the curve $y = \frac{\ln x}{x}$

(i) state the domain of the function **1**

(ii) find the coordinates of the stationary point **2**

(iii) establish the nature of the stationary point **1**

Given that there is a point of inflexion when $x = e^{\frac{3}{2}}$ (no need to find y)

(iv) sketch the curve showing any intercepts with the axes **2**

d) (i) Sketch the curve $y = \log_e(1+x)$ from $x = 0$ to $x = 1$ **1**

(ii) The region bounded by this portion of the curve and the y axis is rotated about the y axis. Find the volume of the solid of revolution so formed. **4**

Mathematics – Assessment Task 3 Year 12 – 2010

Question 1 (16 Marks) **Marks**

a) Find the exact value of $\sec\frac{5\pi}{6}$ **2**

b) A chord in a circle with radius 8 cm subtends an angle of 60° at the centre. Find the exact area of the minor segment cut off by the chord. **3**

c) Solve $2\cos 2x = -1$ for $0 \le x \le 2\pi$ **4**

d) Differentiate:

(i) $x\cos 3x$ **2**

(ii) $\frac{5}{\sin^3 x}$ **2**

e) Evaluate exactly $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sec^2 \frac{x}{2}\, dx$ **3**

Question 2 (17 Marks) Start a New Sheet of paper **Marks**

a) Evaluate exactly $\log_2 \sqrt{8}$ **2**

b) Solve to 3 significant figures $2^x \doteq 27$ **2**

c) Simplify $\frac{\log_c k}{\log_c a} + \log_a k^2$ **2**

d) Differentiate:

(i) e^{-3x} **1**

(ii) $\ln(5x-3)$ **1**

(iii) $\log_e \sqrt{\frac{3x-2}{2x+1}}$ **3**

e) Find $\int \frac{4x^3 - 5x}{x^2}\, dx$ **3**

f) Evaluate in simplest form $\int_2^3 \frac{x}{x^2+1}\, dx$ **3**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

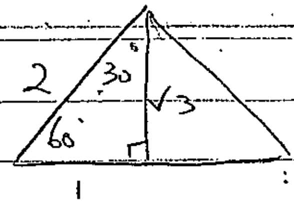
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2010 Mathematics 4SC Task 3

Question 1

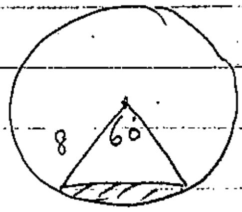
a) $\sec \frac{5\pi}{6} = \sec 150^\circ$



$= -\sec 30^\circ$

$= -\frac{2}{\sqrt{3}}$

b)



$A = \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2} \times 8^2 \left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$

$= 32 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$ cm^2

c) $\cos 2x = -\frac{1}{2}$ $\quad 0 \le 2x \le 4\pi$

$2x$ lies in 2nd or 3rd quads.

Acute angle is $\frac{\pi}{3}$

$\therefore 2x = \pi - \frac{\pi}{3}$ or $\pi + \frac{\pi}{3}$ or $3\pi - \frac{\pi}{3}$ or $3\pi + \frac{\pi}{3}$

$= \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or $\frac{8\pi}{3}$ or $\frac{10\pi}{3}$

$x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$

d) i) $y = x \cos x$

$y' = -3x \sin 3x + \cos 3x$

ii) $y = 5(\sin x)^{-3}$

$y' = -15(\sin x)^{-4} \cos x$

$= \frac{-15 \cos x}{\sin^4 x}$

e) $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sec^2 \frac{x}{2}\, dx = \left[2 \tan \frac{x}{2}\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$

$= 2\left\{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}\right\}$

$= 2\left\{\sqrt{3} - \frac{1}{\sqrt{3}}\right\}$

$= 2\left\{\frac{3-1}{\sqrt{3}}\right\} = \frac{4}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{4\sqrt{3}}{3}$

Question 2

a) Let $\log_2 \sqrt{8} = u$ $\therefore$ $2^u = \sqrt{8} = 2^{3/2}$

$\therefore u = \frac{3}{2}$

$\therefore \log_2 \sqrt{8} = \frac{3}{2}$

b) $x \ln 2 = \ln 27$

$x = \frac{\ln 27}{\ln 2} = 4.7548875$

$x \doteq 4.75$

c) $\log_a k + 2\log_a k = 3\log_a k$

d) i) $y' = -3e^{-3x}$

ii) $y' = \frac{5}{5x-3}$

iii) $y = \frac{1}{2}\{\ln(3x-2) - \ln(2x+1)\}$

$y' = \frac{3}{2(3x-2)} - \frac{1}{2x+1}$

e) $-\frac{e^{-3x}}{3} + c$

f) $\int 4x - \frac{5}{x}\, dx = 2x^2 - 5\ln x + c$

g) $\frac{1}{2}\int_2^3 \frac{2x\, dx}{x^2+1} = \frac{1}{2}\left[\ln(x^2+1)\right]_2^3$

$= \frac{1}{2}(\ln 10 - \ln 5)$

$= \frac{1}{2} \ln \frac{10}{5}$

$= \frac{1}{2} \ln 2$

Question 3

a) $y = 5\sin 2x \qquad x = \frac{\pi}{6},\ y = 5\sin\frac{\pi}{3} = \frac{5\sqrt{3}}{2}$

$\left(\frac{\pi}{6}, \frac{5\sqrt{3}}{2}\right)$

$y' = 10\cos 2x \qquad x = \frac{\pi}{6} \quad m_1 = 10\cos\frac{\pi}{3} = 5$

$\therefore m_2 = \frac{-1}{5}$

$\therefore y - \frac{5\sqrt{3}}{2} = -\frac{1}{5}\left(x - \frac{\pi}{6}\right)$

b) $3\lim_{x\to 0}\frac{\sin 3x}{3x} = 3\times 1 = 3$

c) $y' = -12\sin 3x - 6\cos 3x$

$y'' = -36\cos 3x + 18\sin 3x$

$= -9(4\cos 3x - 2\sin 3x)$

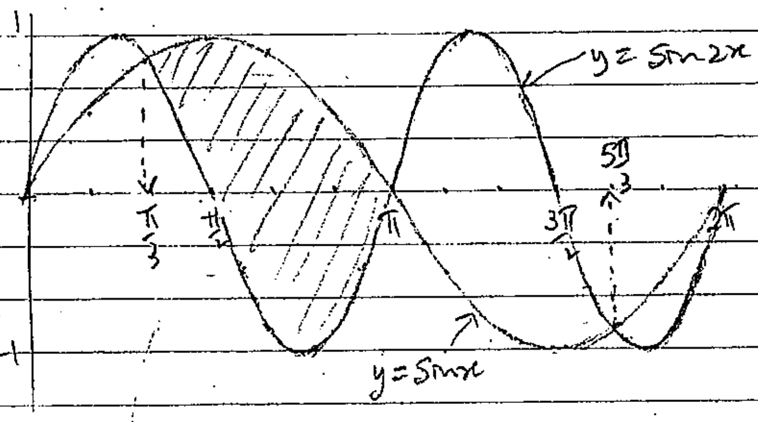
$= -9y$

d) i) $\sin 2\times\frac{\pi}{3} = \sin 120^\circ = +\sin 60^\circ = \frac{\sqrt{3}}{2}$

$\sin\frac{\pi}{3} = \sin 60^\circ = \frac{\sqrt{3}}{2}$

$\therefore x = \frac{\pi}{3}$ is a solution

ii)



iii) Other solutions are $x = \pi$ & $x = 2\pi - \frac{\pi}{3}$ ie $x = \frac{5\pi}{3}$

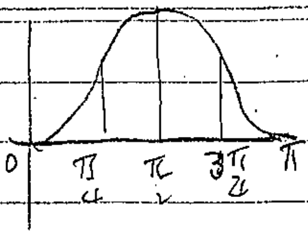
iv) Area $= \int_{\frac{\pi}{3}}^{\pi} \sin x - \sin 2x \, dx = \left[-\cos x + \frac{1}{2}\cos 2x\right]_{\pi/3}^{\pi}$

$= -\cos\pi + \frac{1}{2}\cos\frac{\pi}{3} + \cos\frac{\pi}{3} - \frac{1}{2}\cos\frac{2\pi}{3}$

$= -(-1) + \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2} - \frac{1}{2}\left(-\frac{1}{2}\right)$

$= 1 + \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 2u^2$

c) $h = \frac{\pi}{4}$



x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$f(x)$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0

$$I \doteq \frac{h}{3}\{y_0 + 4y_1 + y_2\} + \frac{h}{3}\{y_2 + 4y_3 + y$$

$$I \doteq \frac{h}{3}\{y_0 + y_4 + 4(y_1 + y_3) + 2y_2\}$$

$$I \doteq \frac{\pi}{12}\{0 + 0 + 4(\tfrac{1}{2} + \tfrac{1}{2}) + 2 \times 1\}$$

$$I \doteq \frac{\pi}{12} \times 6$$

$$I \doteq \frac{\pi}{2}$$

Question 4

a) i) $y = xe^{2x}$

$$y' = x.2e^{2x} + e^{2x}.1$$

$$y' = 2xe^{2x} + e^{2x}$$

ii)

$$\therefore \int 2xe^{2x} + e^{2x}\,dx = xe^{2x}$$

$$2\int xe^{2x}\,dx + \int e^{2x}\,dx = xe^{2x}$$

$$2\int xe^{2x}\,dx = xe^{2x} - \int e^{2x}\,dx$$

$$= xe^{2x} - \frac{e^{2x}}{2}$$

$$\int xe^{2x}\,dx = \frac{x}{2}e^{2x} - \frac{e^{2x}}{4} + c$$

c) $y = \dfrac{\ln x}{x}$

i) $D: \ x > 0$ (for $\ln x$ to exist)

ii) $y' = \dfrac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}$

$= \dfrac{1 - \ln x}{x^2}$

For Stat pt $y' = 0$ $\qquad 1 - \ln x = 0$

$\ln x = 1$

$x = e$

$y = \frac{1}{e}$

Stat pt at $(e, \frac{1}{e})$

iii)

x	e^-	e	e^+
y'	$+$	0	$-$

using y'

$\therefore$ $(e, \frac{1}{e})$ is a maximum turning point

iv) For points of inflexion $y'' = 0$

$y'' = \dfrac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x) \cdot 2x}{x^4}$

$= \dfrac{-x - 2x + 2x\ln x}{x^4}$

$= \dfrac{-3x + 2x\ln x}{x^4} = \dfrac{-3 + 2\ln x}{x^3}$

Pt of inflexion when

$$-3 + 2\ln x = 0$$

$$2\ln x = 3$$

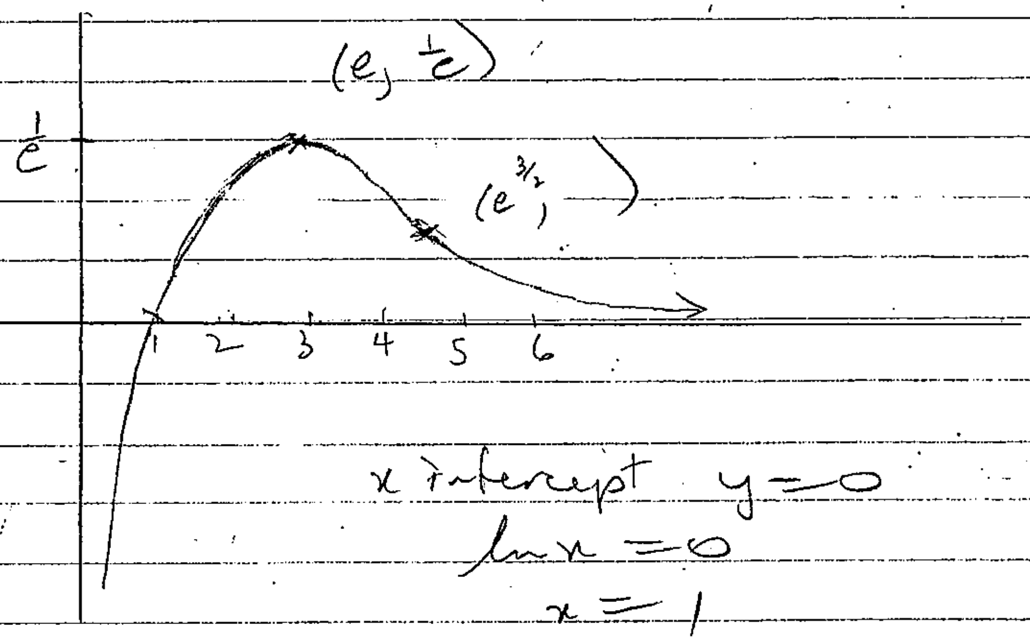
$$\ln x = \frac{3}{2}$$

$$x = e^{3/2}$$

Show y'' changes sign

x	$e^{3/2^-}$	$e^{3/2}$	$e^{3/2^+}$
y''	$-$	0	$+$

Concavity changes from concave down to concave up

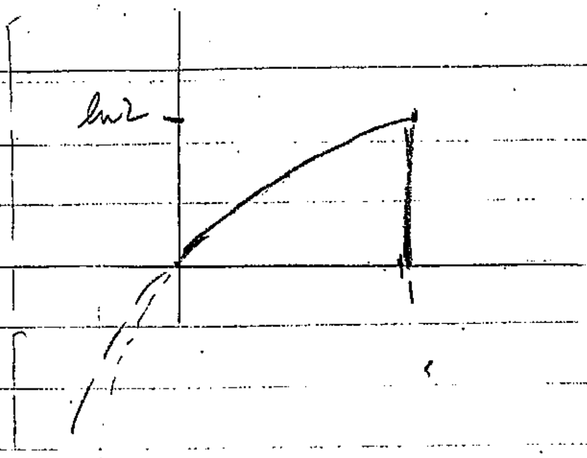


x intercept $y = 0$

$$\ln x = 0$$

$$x = 1$$

d)



$y = \log_e(1+x)$

$1+x = e^y$

$x = e^y - 1$

$x^2 = e^{2y} - 2e^y + 1$

$$V = \pi \int x^2 \, dy$$

$$= \pi \int_0^{\ln 2} e^{2y} - 2e^y + 1 \, dy$$

$$= \pi \left[\frac{e^{2y}}{2} - 2e^y + y \right]_0^{\ln 2}$$

$$= \pi \left\{ \left(\frac{e^{2\ln 2}}{2} - 2e^{\ln 2} + \ln 2 \right) - \left(\frac{1}{2} - 2 + 0 \right) \right\}$$

$$= \pi \left\{ \left(\frac{4}{2} - 2 \times 2 + \ln 2 \right) - -\frac{3}{2} \right\}$$

$$= \pi \left[2 - 4 + \ln 2 + \frac{3}{2} \right]$$

$$= \pi \left(\ln 2 - \frac{1}{2} \right) u^3 = \frac{\pi}{2} (2\ln 2 - 1) u^3$$