HSC Mathematics
Assessment Task 3
June 2011

- Time allowed: 60 minutes plus 5 minutes reading time
- Answer each question on your own paper showing all necessary working.
- Start each question on a new page.
- Marks may not be awarded for untidy or poorly set out work.

Topic	Mark
Trigonometric Functions	/25
Applications of Calculus to the Physical World	/30

TOTAL **/55**

TRIGONOMETRIC FUNCTIONS

Question 1.			**Marks**
a)	Express $\frac{2\pi}{3}$ radians in degrees.		1
b)	The angle subtended at the centre of a circle of radius 8 cm is 1.2 radians. Find:		
	i)	the length of the arc cut off by this angle	1
	ii)	the area of the sector formed.	1
c)	Differentiate with respect to x:		
	i)	$\sin 3x$	1
	ii)	$\tan \pi x$	1
	iii)	$x \cos x$	2
	iv)	$\sin^2 x$	2
d)	Find:		
	i)	$\int \sin x \, dx$	1
	ii)	$\int \cos\left(2x + \frac{\pi}{3}\right) dx$	1
	iii)	$\int \sec 2x \tan 2x \, \mathrm{dx}$	1
	iv)	$\int \frac{\sin x}{1 + \cos x} \, dx$	2

Question 2. START A NEW PAGE

a) For the function $y = 3\sin 2x$, find:

i) its amplitude **1**

ii) its period **1**

b) Solve $2\sin^2 x = \sin x$ for $0 \le x \le 2\pi$ **2**

c) **i)** On the same diagram, draw a neat sketch of $y = \tan \pi x$ and $y = 1 - x$ for $0 \le x \le 2$ **3**

ii) Hence determine the number of solutions to the equation $\tan \pi x = 1 - x$ for $0 \le x \le 2$. **1**

d) The area under the curve $y = \sec x$, for $0 \le x \le \frac{\pi}{4}$ is rotated about the x axis. What is the volume of the solid of revolution generated? **3**

APPLICATIONS OF CALCULUS TO THE PHYSICAL WORLD

Question 3. START A NEW PAGE

a) A stone dropped into a still pond causes circular ripples on the surface. The area of disturbed water, t seconds after the stone hits the surface of the water is given by $A = 4\pi t^2$

i) Find an expression to represent the rate of change of the area of disturbed water. **1**

ii) Find, in terms of π, the rate of change of the area of the disturbed water after 1 second. **1**

b) A particle moves on the x axis with velocity after t seconds given by $v = 6 - 2t$ m/s. Initially it is at $x=4$.

i) Find its position as a function of t. **2**

ii) When is the particle at rest? **1**

iii) Find the total distance travelled in the first 7 seconds. **2**

c) Water enters a container, initially empty, so that, after t minutes the volume V L of water in it is increasing at a rate of $\dfrac{12t}{t^2+4}$ L/min.

i) Show that $V = 6\ln\left(\dfrac{t^2+4}{4}\right)$ **2**

ii) Find, to the nearest minute, the time taken for the container to hold 17 L. **2**

d) The population of a colony of insects at time t years is given by $P = 40e^{\frac{1}{2}t}$.

i) Show that the growth is exponential. **1**

ii) What is the initial population? **1**

iii) What is the population after 10 years? **1**

iv) What is the rate of increase of the population after 10 years? **1**

Question 4. START A NEW PAGE

a) A particle P is moving along the x axis. Its position at time t seconds is given by

$$x = 2\sin t - t, \qquad t \geq 0.$$

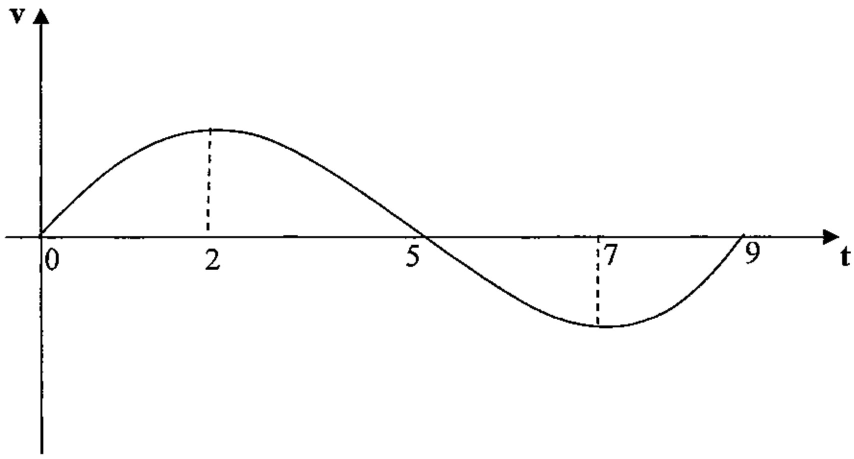
i) Find an expression for the velocity of the particle. **1**

ii) In what direction is the particle moving at $t = 0$? **1**

iii) Determine when the particle first comes to rest. **1**

iv) When is the acceleration negative for $0 \leq t \leq 2\pi$? **2**

b)



The above graph shows the velocity, v m/s, of a particle moving on a straight line, for $0 \leq t \leq 9$.

State all the times, or intervals of time, for which

i) the particle is stationary **1**

ii) the particle is moving in the positive direction **1**

iii) the acceleration is positive **1**

iv) the particle is slowing down. **1**

c) The mass, M g, of a radioactive element present in a substance after t years is given by $M = M_0 e^{-kt}$, where M_0 is the initial mass and k is a constant. The half-life period of the element is 100 years.

i) Show that $k = \dfrac{\ln 2}{100}$ **2**

ii) How long will it take for 9 g to reduce to 2 g? **2**

iii) What percentage of an original mass will be present after 32 years? **2**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \ x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \ x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \ a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \ x > 0$

2011 HSC 2u Task 3

Question 1

a) $\frac{2\pi}{3} = \frac{2\times180}{3} = 120^\circ$

b) i) $l = r\theta$

$l = 8\times1.2$

$= 9.6$ cm

ii) $A = \frac{1}{2}r^2\theta$

$= \frac{1}{2}\times8^2\times1.2$

$= 38.4$ cm^2

iv) $\int \frac{\sin x\,dx}{1+\cos x}$

$= -\ln(1+\cos x) + c$

c) i) $y = \sin 3x$

$y' = 3\cos 3x$

ii) $y = \tan \pi x$

$y' = \pi \sec^2 \pi x$

iii) $y = x\cos x$

$y' = -x.\sin x + \cos x$

iv) $y = \sin^2 x$

$y' = 2\sin x\cos x$

d) i) $\int \sin x\,dx = -\cos x + c$

ii) $\int \cos\left(2x+\frac{\pi}{3}\right)dx = \frac{1}{2}\sin\left(2x+\frac{\pi}{3}\right) + c$

iii) $\int \sec 2x \tan 2x\,dx = \frac{1}{2}\sec 2x + c$

2

a) $y = 3\sin 2x$

i) $a = 3$

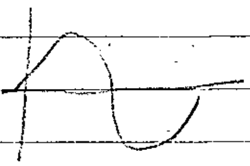
ii) $T = \frac{2\pi}{2} = \pi$

b) $2\sin^2 x = \sin x$

$2\sin^2 x - \sin x = 0$

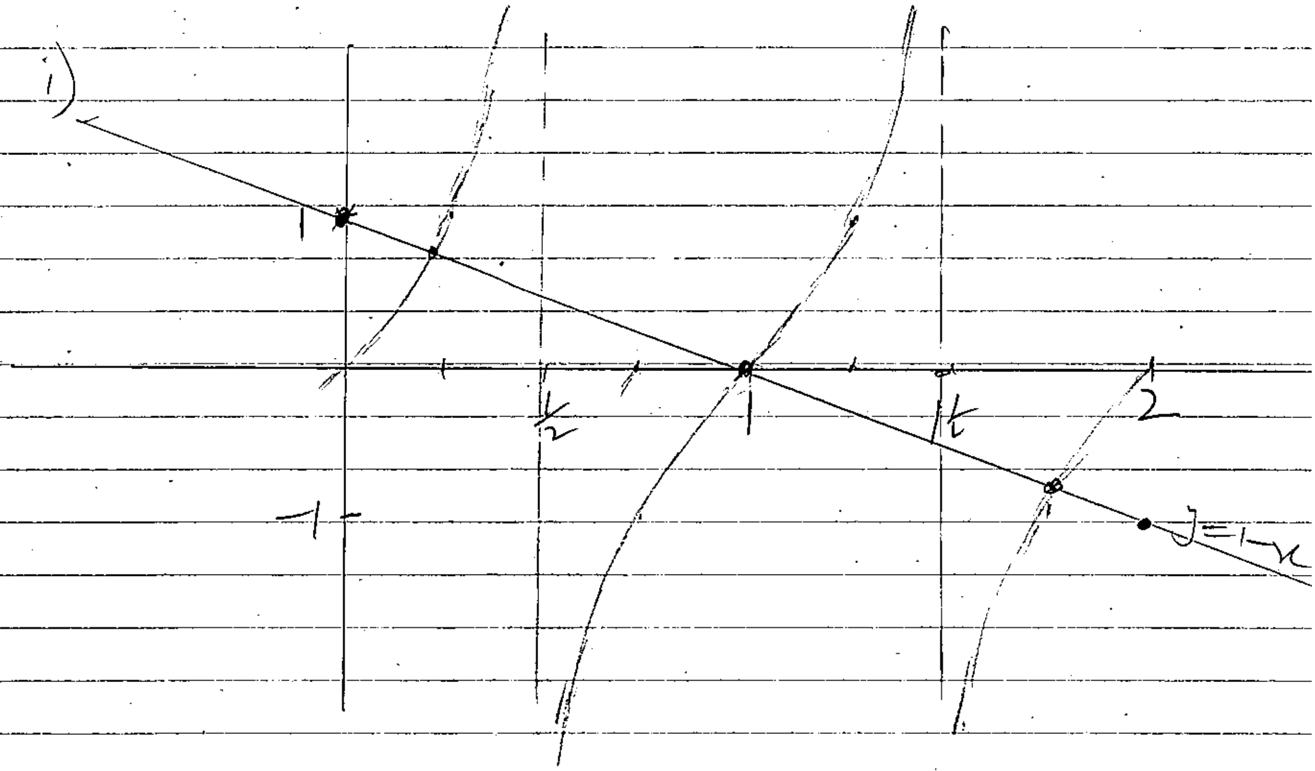
$\sin x\,(2\sin x - 1) = 0$

$\sin x = 0$ or $\sin x = \frac{1}{2}$



$x = 0, \pi, 2\pi$ $\quad$ $x = \frac{\pi}{6}, \frac{5\pi}{6}$

c) i)



ii) there are 3 solutions.

d) $V = \pi \int_0^{\frac{\pi}{4}} \sec^2 x \, dx = \pi \left[\tan x\right]_0^{\pi/4}$

$= \pi \times 1 - \pi \times 0 = \pi u^3$

Question 3

a) $A = 4\pi t^2$

i) $\frac{dA}{dt} = \underline{\underline{8\pi t}}$ 1

ii) $t = 1$, $\frac{dA}{dt} = \underline{\underline{8\pi}}\ u^2/s$ 1

b) $v = 6 - 2t$

i) $x = 6t - t^2 + c$

$t = 0$, $x = 4$

$4 = 0 - 0 + c$

$\boxed{\therefore x = 6t - t^2 + 4}$ 2

ii) $v = 0$ for at rest

$0 = 6 - 2t$ 1

$\underline{\underline{t = 3}}$

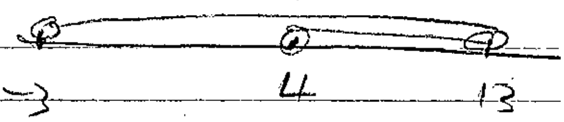
At 3 secs particle is at rest.

iii) Dist $t = 0$ $x = 4$

$t = 7$ $x = -3$ 2

When $t = 3$, $x = 13$

Distance travelled:



$13 - 4 = 9$

$d = 2 \times 9 + 7$

$= \underline{\underline{25 \text{ metres.}}}$

3 c)

$$\frac{dV}{dt} = \frac{12t}{t^2+4}$$

$$V = \int \frac{12t}{t^2+4} \, dt$$

$$V = 6\ln(t^2+4) + c$$

Initially $t=0 \Rightarrow V=0$

2

$$0 = 6\ln 4 + c$$

$$c = -6\ln 4$$

$$V = 6\ln(t^2+4) - 6\ln 4.$$

$$V = 6\ln\left(\frac{t^2+4}{4}\right)$$ as required.

ii)

$$17 = 6\ln\left(\frac{t^2+4}{4}\right)$$

2

$$\frac{17}{6} = \ln \frac{t^2+4}{4}$$

$$t^2+4 = 4e^{17/6} \Rightarrow t^2 = 4e^{\frac{17}{6}} - 4$$

$$t = +\sqrt{64.008} \doteqdot 8 \text{ mins}$$

d i)

$$P = 40e^{t/2}$$

$$\frac{dP}{dt} = 40 \times \frac{1}{2} e^{t/2}$$

$$= 20e^{\frac{t}{2}}$$

1

$$= \frac{1}{2}P$$

ie rate proportional to P

ii) $t=0 \quad P = 40e^0 = 40$

1

iii) $t = 10$

$P = 40e^{\frac{1}{2} \times 10}$

$= 5936.52$

$P \doteqdot 5937$ 1

iv) $\frac{dP}{dt} = 40 \times \frac{1}{2} e^{\frac{1}{2}t}$

$= \frac{1}{2}\left(40 \times e^{5}\right)$

$= 2968$ insects / year

1

Question 4

a) $x = 2\sin t - t$

i) $v = 2\cos t - 1$ 1

ii) $t = 0$, $v = 2 - 1$

$= 1$ 1

moving to the right

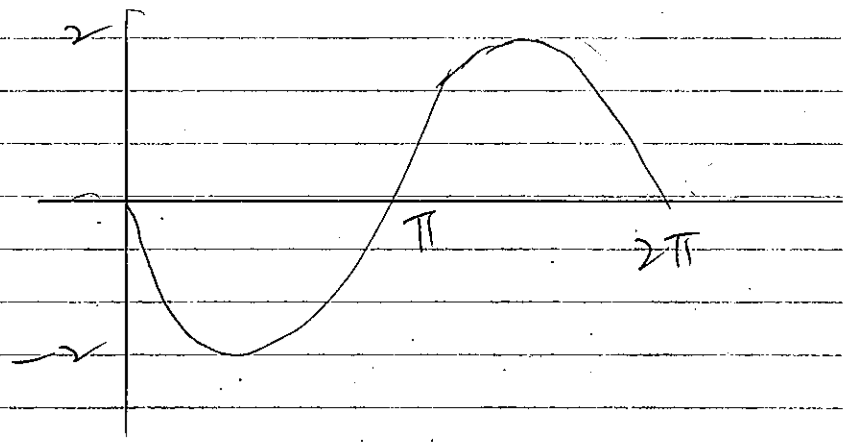
iii)

$2\cos t - 1 = 0$

$\cos t = \frac{1}{2}$ 1

$t = \frac{\pi}{3}$, — — — —

iv) $\ddot{x} = -2\sin t$



Acc is negative for 2

$0 < t < \pi$

b) i) Stationary $t = 0, 5, 9$ 1

ii) $0 < t < 5$ 1

iii) Gradient of v graph is positive 1

$0 < t < 2$ and $7 < t < 9$

iv) v & $\ddot{x}$ in opposite directions

$2 < t < 5$ and $7 < t < 9$ 1

$v > 0$, $\ddot{x} < 0$ $v < 0$ $\ddot{x} > 0$

c) $$M = M_0 e^{-kt}$$

$t = 100$, $M = \frac{M_0}{2}$

$$\frac{M_0}{2} = M_0 e^{-100k}$$

$$0.5 = e^{-100k}$$

$$2 = e^{100k}$$

$$100k = \ln 2$$

$$k = \frac{\ln 2}{100}$$ 2

ii) $$M = M_0 e^{-\frac{\ln 2}{100} t}$$

$$2 = 9 e^{-\frac{\ln 2}{100} t}$$

$$\ln \frac{2}{9} = -\frac{\ln 2}{100} t$$

$$t = \ln 2/9 \div \left(-\frac{\ln 2}{100}\right)$$

$$t = \frac{\ln \frac{2}{9}}{-\frac{\ln 2}{100}}$$

$$= 216.99$$

$$t = 217 \text{ years}$$ 2

iii)

$$M = M_0 e^{-\frac{\ln 2}{100} \times 32}$$

$$M = 0.801 M_0$$

80.1% of original mass is present after 32 years. ✓

80.1