# Gosford High School

# Year 12

## 2012

# Higher School Certificate

# Mathematics

# Assessment Task 3

## Time Allowed – 60 minutes

(reading time 5 minutes)

Remember to use the **provided multiple choice answer sheet** to answer **Question1**
Remember to start each new question on a **new page for Questions 2 to 5**
Students must answer questions using a blue/black pen and/or a sharpened B or HB pencil.
Approved scientific calculators may be used
Students need to be aware that

* 'bald' answers may not gain full marks.
* untidy and/or poorly organised solutions may not gain full marks.

**QUESTION 1** ***(please use the provided multiple choice answer sheet)*** **5 marks**

i) $e^{2\ln x} = ?$

A) $2\ln x$ B) $2x$

C) $2^x$ D) $x^2$

ii) The $\int xe^{x^2}\,dx$ is equal to

A) $\frac{1}{2}e^{x^2} + c$ B) $2e^{x^2} + c$

C) $\frac{e^{x^3}}{3} + c$ D) None of these

iii) The graph of $y = \cos\left(x + \frac{\pi}{2}\right)$ for all values of $x$ is the same as the graph of :-

A) $y = \sin x$ B) $y = \cos x$

C) $y = -\sin x$ D) $y = -\cos x$

iv) If $x$ is an extremely small positive value, which one of the following is NOT true

A) $\cos x \doteqdot 1$ B) $x \doteqdot \sin x \doteqdot \tan x$

C) $\sin x < x < \tan x$ D) $\tan x < x < \sin x$

v) If $A = B + Ce^{-kt}$ then

A) $t = \frac{1}{k}\log_e\left[\frac{A-B}{C}\right]$ B) $t = \frac{1}{k}\log_e\left[\frac{B-A}{C}\right]$

C) $t = \frac{1}{k}\log_e\left[\frac{C}{A-B}\right]$ D) $t = -\log_e\left[\frac{A-B}{kC}\right]$

**QUESTION 2** *start a new page* **(11 marks)**

a) Find $\dfrac{d}{dx}\left[e^{x^2-2x+7}\right]$ (1)

b) Find primitives of (i) $\dfrac{1}{7-2x}$ (1)

(ii) $e^{4-3x}$ (1)

c) Evaluate $\displaystyle\int_{-1}^{1}\frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$ (2)

d) If $y=\log_e\left(\dfrac{1+x}{1-x}\right)$ show that $\dfrac{dy}{dx}=\dfrac{2}{1-x^2}$ (3)

e) The portion of the curve $y=e^x+2$ from $x=0$ to $x=1$ is rotated about the $x$ axis.

Find the volume of the solid generated. (3)

**QUESTION 3** ***start a new page*** **(10 marks)**

a) Find $\int \sin(2x)\,dx$ (1)

b) State the amplitude and period of the curve $y = -2\cos\left(\frac{\pi x}{2}\right)$ (2)

c) Find $\frac{d}{dx}\left[\,\cos(3x)\,\right]$ (1)

d) If $y = \sin^2\left(\frac{x}{2}\right)$, find $\frac{dy}{dx}$ (1)

e) Find the equation of the tangent at $x = \frac{\pi}{6}$ on the curve $y = \tan(2x)$, giving

your answer in simplest exact form. (3)

f) Consider the function $f(x) = 1 - 2\sin(x + \frac{\pi}{4})$ for $0 \le x \le 2\pi$

(i) Without using calculus, state the maximum value of $f(x)$. (1)

(ii) State the value of $x$, in the given domain, for which this maximum occurs. (1)

**QUESTION 4** *start a new page* **(11 marks)**

a) Find the value of $\log_7 43700$ correct to 3 significant figures. (1)

b) Sketch the curve $y = \log_e(x+2)$, carefully labelling the intercepts on the coordinate axes and any asymptotes. (2)

c) Consider $f(x) = x - 3\ln x$

Find the coordinates of the stationary point on the curve $y = f(x)$ and determine its nature. (4)

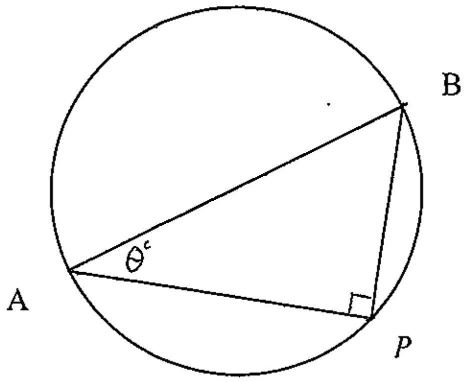
d) Show that the curve $y = xe^{-x}$ has a point of inflexion at $x = 2$ (4)

**QUESTION 5** *start a new page* **(13 marks)**

a) The gradient of the tangent at a point $(x, y)$ on the curve $y = f(x)$ is given by

$\cos(2x) - \sec^2 x$. If the curve passes through the point $\left(\frac{\pi}{4}, \frac{1}{2}\right)$, find $f(x)$. (2)

b) A point P moves along the circumference of a circle with radius R and diameter AB. If $\angle APB = \frac{\pi}{2}^c$ and $\angle PAB = \theta^c$.



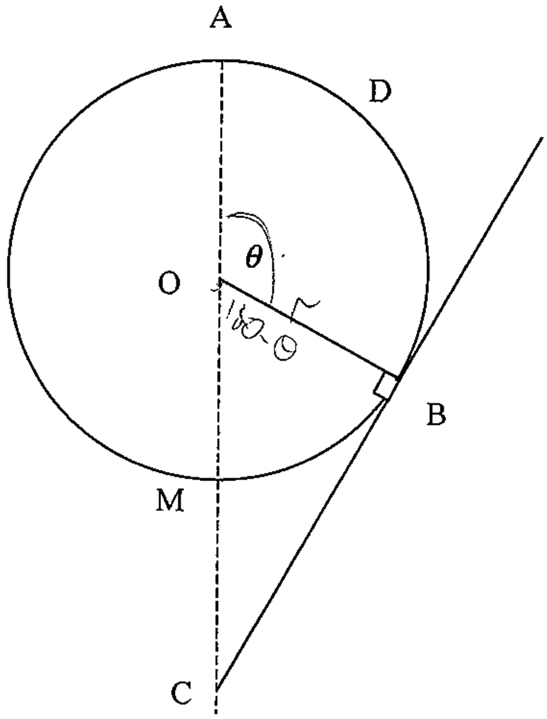

(i) Show that $PA = 2R\cos\theta$. (1)

(ii) Hence show that the area of $\Delta PAB = 2R^2 \sin\theta\cos\theta$ (1)

(iii) Given the identity that $2\sin\theta\cos\theta = \sin 2\theta$, the above area (A) of $\Delta PAB$ can be written as $A = R^2 \sin 2\theta$.

Show that the area of $\Delta PAB$ is a maximum when $\theta = \frac{\pi}{4}^c$ (3)

c)




In the diagram, O is the centre of the circle with radius 'r'.

A, O, M and C are collinear. Given Arc ADB + interval BC = $l$

(i) Show that $\theta - \tan\theta = \frac{l}{r}$. (3)

(ii) If the area of the minor sector AOB is $\frac{3}{8}$ of the area of the circle,

find $\theta$ and ratio $\frac{l}{r}$. (3)

**END OF EXAMINATION**

MULTIPLIE CHOICE ANSWER SHEET

YEAR 12 MATHEMATICS ASSESSMENT TASK 3

QUESTION 1

| | | | | |
|---|---|---|---|---|
| (i) | A | B | C | D |
| (ii) | A | B | C | D |
| (iii) | A | B | C | D |
| (iv) | A | B | C | D |
| (v) | A | B | C | D |

# STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

# YEAR 12 HSC MATHEMATICS 2012

## SOLUTIONS

**Question 1** (i) D (ii) A

(iii) C (iv) D

(v) C

**Question 2** a) $(2x-2)\,e^{x^2-2x+7}$ (1)

b) (i) $-\frac{1}{2}\log_e(7-2x)+c$ (1)

(ii) $-\frac{1}{3}e^{4-3x}+c$ (1)

c) $$\int_{-1}^{1}\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\,dx = \left[\log_e(e^{x}+e^{-x})\right]_{-1}^{1} \quad (1)$$

$$= \log_e(e+e^{-1}) - \log_e(e^{-1}+e)$$

$$= 0 \quad (1)$$

d) $$y = \log_e(1+x) - \log_e(1-x)$$

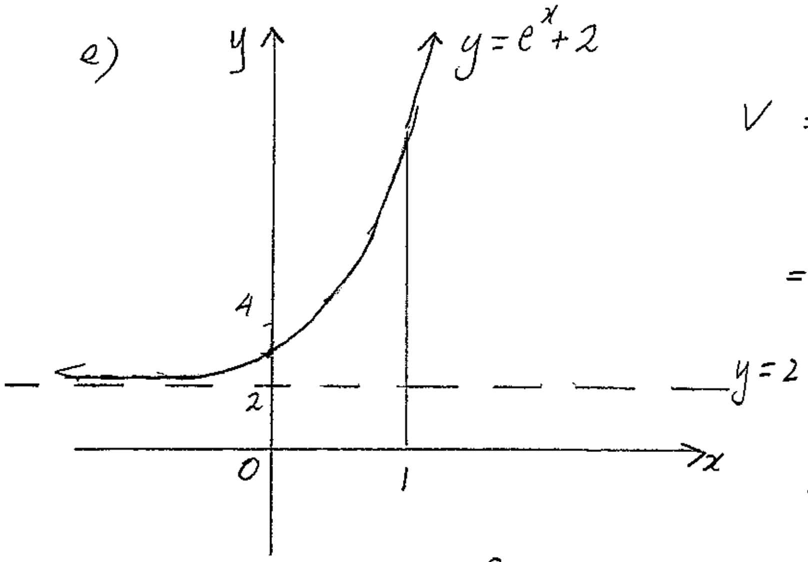
$$= \frac{1}{1+x} - \frac{-1}{1-x} \quad (2)$$

$$= \frac{1}{1+x} + \frac{1}{1-x}$$

$$= \frac{1(1-x)+1(1+x)}{(1+x)(1-x)}$$

$$= \frac{2}{1-x^2} \quad (1)$$

e)




$$V = \pi\int_0^1 (e^x+2)^2\,dx \quad (1)$$

$$= \pi\int_0^1 (e^{2x}+4e^x+4)\,dx.$$

$$= \pi\left[\tfrac{1}{2}e^{2x}+4e^x+4x\right]_0^1 \quad (1)$$

$$\therefore V = \pi\left[\tfrac{1}{2}e^2+4e+4-\left(\tfrac{1}{2}+4+0\right)\right]$$

$$= \pi\left[\tfrac{1}{2}e^2+4e-\tfrac{1}{2}\right] \text{ cubic units} \quad (1)$$

<u>Question 3</u>

a) $\int \sin(2x)\,dx = -\frac{1}{2}\cos(2x)+c \quad (1)$

b) Amplitude $= 2$, Period $= \frac{2\pi}{\pi/2}$

$$= 4 \quad (2)$$

c) $\frac{d}{dx}\left[\cos(3x)\right] = -3\sin(3x)$

d) $y = \left[\sin\left(\frac{x}{2}\right)\right]^2$

$$\frac{dy}{dx} = 2\left[\sin\left(\frac{x}{2}\right)\right]^1 \cdot \frac{1}{2}\cos\left(\frac{x}{2}\right) \quad (1)$$

$$= \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

e) $y = \tan(2x)$ when $x = \frac{\pi}{6}$, $y = \tan\left(\frac{\pi}{3}\right)$
$= \sqrt{3}$

$\frac{dy}{dx} = 2\sec^2(2x)$ when $x = \frac{\pi}{6}$, $\frac{dy}{dx} = 2\sec^2\left(\frac{\pi}{3}\right)$
$= 8$ (2)

Equation of Tangent $y - \sqrt{3} = 8\left(x - \frac{\pi}{6}\right)$ (1)

$y = 8x - \frac{4\pi}{3} + \sqrt{3}$

f) (i) $-1 \le \sin\left(x + \frac{\pi}{4}\right) \le 1$.

$\therefore$ Max. Value of $f(x)$ is 3 (1)

(ii) Solving $\sin\left(x + \frac{\pi}{4}\right) = -1$

$x + \frac{\pi}{4} = \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

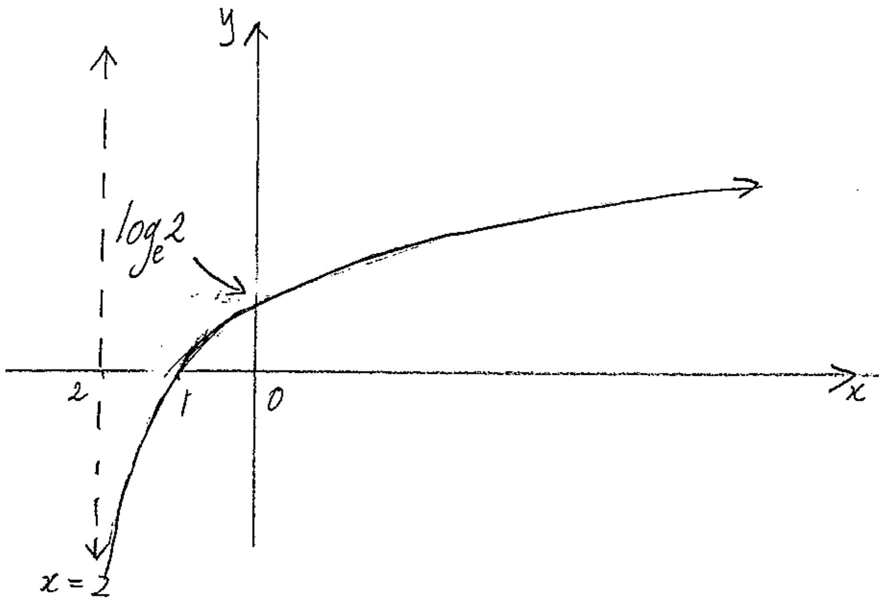
$x = \frac{5\pi}{4}, \frac{9\pi}{4}$ ← outside domain.

$\therefore$ When $x = \frac{5\pi}{4}$ (1)

<u>Question 4</u> a) $\log_7 43700 = \frac{\ln 43700}{\ln 7}$ (1)

$=$

b)




(2)

c) $f(x) = x - 3\ln x$

$f'(x) = 1 - \frac{3}{x}$

(1)

$= 1 - 3x^{-1}$

$f''(x) = 3x^{-2}$

$= \frac{3}{x^2} > 0$

for all $x$:

Always concave up

(1)

For stationary pts

$f''(x) = 0$

$\therefore 1 - \frac{3}{x} = 0$

$x = 3$ , $y = 3 - 3\ln 3$

$\therefore (3, 3 - 3\ln 3)$ is a minimum turning pt.

(1)

d)

$$y = xe^{-x}$$

$$\frac{dy}{dx} = e^{-x} \cdot 1 + x \cdot (-e^{-x})$$

$$= e^{-x} - xe^{-x} \qquad (1)$$

$$\frac{d^2y}{dx^2} = -e^{x} - \left[ e^{x} - xe^{-x} \right]$$

$$= -2e^{-x} + xe^{-x}$$

$$= e^{-x}(x-2) \qquad (2)$$

$$= 0 \quad \text{when } x = 2$$

$\therefore$ Possible pt of inflexion at $x = 2$.

When $x < 2$, $\frac{d^2y}{dx^2} = (+)(-) < 0$

when $x > 2$, $\frac{d^2y}{dx^2} = (+)(+) > 0$ (1)

$\therefore$ Concavity Change

$\therefore$ Pt. of Inflexion at $x = 2$

Question 5

a) $f'(x) = \cos(2x) - \sec^2 x$

$$f(x) = \frac{1}{2}\sin(2x) - \tan x + c \qquad (1)$$

$$f\left(\frac{\pi}{4}\right) = \frac{1}{2} - 1 + c$$

$$\frac{1}{2} = \frac{1}{2} - 1 + c$$

$$c = 1$$

$$\therefore f(x) = \frac{1}{2}\sin(2x) - \tan x + 1 \qquad (1)$$

b) (i) $\cos\theta = \dfrac{AP}{AB}$

$\cos\theta = \dfrac{AP}{2R}$

$\therefore AP = 2R\cos\theta$ (1)

(ii) $\text{Area} = \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 2R \times 2R\cos\theta \cdot \sin\theta$

$= 2R^2 \sin\theta\cos\theta$ (1)

(iii) $A = R^2 \sin 2\theta$ using given substitution

$\dfrac{dA}{d\theta} = 2R^2\cos 2\theta$

$= 0$ when $\theta = \dfrac{\pi}{4}$ (1)

$\dfrac{d^2A}{d\theta^2} = -4R^2 \sin 2\theta$

$= -4R^2 < 0$ when $\theta = \dfrac{\pi}{4}$ (2)

$\therefore$ Maximum Area occurs when $\theta = \dfrac{\pi}{4}$.

c) (i) Arc ADB $= r\theta$ using $l = r\theta$. (1)

In $\Delta OBC$, $\tan(180 - \theta) = \frac{BC}{r}$

$$-\tan\theta = \frac{BC}{r}$$

$$BC = -r\tan\theta \quad (1)$$

$\therefore$ Arc ADB + Interval BC $= r\theta - r\tan\theta$.

$$l = r(\theta - \tan\theta)$$

$$\frac{l}{r} = \theta - \tan\theta \quad (1)$$

(ii) $\frac{1}{2}r^2\theta = \frac{3}{8}\pi r^2$ (1)

$$\theta = \frac{3\pi}{4} \quad (1)$$

$$\therefore \frac{l}{r} = \frac{3\pi}{4} - \tan\frac{3\pi}{4}$$

$$= \frac{3\pi}{4} + 1 \quad (1)$$