GOSFORD HIGH SCHOOL

HSC MATHEMATICS

2015

Task 3

General Instructions:

Reading time: 5 minutes

Working time: 60 minutes

Answer the multiple choice questions on the answer sheet provided.

Start questions 5, 6 and 7 each on a new page. All necessary working should be shown for every question.

MC	/2	/2
Q5	/16	
Q6		/15
Q7	/7	/10
	Trigonometry /25	Logarithms and Exponentials /27
Total		/52

Multiple choice Answer on the answer sheet provided.

1) If $\log_4 x = 7$, then $x =$

(A) 4^7

(B) 7^4

(C) $\sqrt[4]{7}$

(D) $\sqrt[7]{4}$

2) Evaluate

$$\int_0^1 e^{2x}\,dx$$

(A) e^2

(B) $2(e^2 - 1)$

(C) $\frac{e^2-1}{2}$

(D) $\frac{e^2}{2} - 1$

3) $$\int \frac{\cos 2x}{3 + \sin 2x}\,dx$$ is equivalent to:

(A) $\frac{1}{2}\ln(3 + \sin 2x) + C$

(B) $-2\ln(3 + \cos 2x) + C$

(C) $-\frac{2(1+3\sin(2x))}{(3+\sin 2x)^2} + C$

(D) $\frac{2x}{3} + C$

4) How many solutions of the equation $\cos 2x\,(1 - \sin x) = 0$ lie between 0 and 2π?

(A) 3

(B) 4

(C) 5

(D) 6

Question 5 Start on a new page

a) Convert $40°$ to radians. 1

b) Differentiate with respect to x:

(i) $\tan x$ 1

(ii) $e^{\sin x}$ 1

(iii) $x \sin x$ 1

(iv) $\cos^2 3x$ 2

c) Find: 1

$$\int \cos 2x \ dx$$

d) For the function $y = 1 + 2\sin 2x$

(i) State the period 1

(ii) State the amplitude 1

(iii) Draw a neat sketch of $y = 1 + 2\sin 2x$ in the domain $0 \leq x \leq 2\pi$. 2

e) Determine the exact value of: 3

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 2\sec^2 x \ dx$$

f) Determine the exact area of a segment subtending an angle of $\frac{2\pi}{3}$ radians at the centre of a circle of radius $2\sqrt{3}$ units. 2

Question 6 Start on a new page.

a) State the domain of the function $y = \log_e(x-3)$ 1

b) Differentiate with respect to x.

(i) $\ln(2x+3)$ 1

(ii) $\ln(x-5)^7$ 1

c) Determine the exact value of 2

$$\int_0^3 x^2 - e^x \, dx$$

d) (i) Evaluate $\log_5 1732$ correct to 3 significant figures. 1

(ii) If $\log_a x = 0.7$ and $\log_a y = 0.4$, evaluate $\log_a \frac{x^3}{\sqrt{y}}$ 2

e) The gradient of a curve at any point on it is $\frac{2}{2x+1}$ and the curve passes through $(1, \ln 3)$ 3
Find the equation of the curve.

f) Consider the function $f(x) = e^x - x$.

(i) Find $f'(x)$. 1

(ii) Find the coordinates and the nature of the stationary point on the curve $y = f(x)$. 3

Question 7 Start on a new page

a) Find the volume when the region bound by $y = \tan x$, $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x axis. 4

b) Find all the solutions of $2\cos^2 x + \sin x - 2 = 0$ where $0 \leq x \leq 2\pi$ 3

c) Find:

(i) $\displaystyle\int \frac{x}{x^2 - 3}\, dx$ 2

(ii) $\displaystyle\int \frac{x^2 - 3}{x}\, dx$ 2

d) The tangent to the curve $y = \ln x$ at point $P(x_1, y_1)$ has the equation $y = mx$.

(i) Sketch the line and the curve on one graph. 2

(ii) Show that $m = \frac{1}{x_1}$ 1

(iii) Show that the coordinates of P are $(e, 1)$. 2

(iv) Determine the exact value of m. 1

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

HSC Mathematics Task 3 2015

Student Name/Number:____________________________

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2+4=$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○

HSC Mathematics Task 3 2015

1. A
2. C
3. A
4. C

5. a) $40 \times \frac{\pi}{180} = \frac{2\pi}{9}^{c}$

b) (i) $\frac{d}{dx} \tan x = \sec^2 x$

(ii) $\frac{d}{dx} e^{\sin x} = \cos x \, e^{\sin x}$

(iii) $\frac{d}{dx} x \sin x = x \cos x + \sin x$

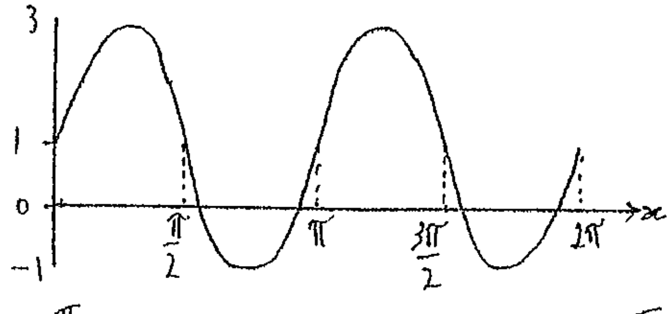
(iv) $\frac{d}{dx} \cos^2 3x = -6 \cos 3x \sin 3x$

c) $\int \cos 2x \, dx = \frac{1}{2} \sin 2x + C$

d) (i) $\frac{2\pi}{2} = \pi$

(ii) 2

(iii)



e) ~~(iii)~~ $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 2\sec^2 x \, dx = 2\left[\tan x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= 2\left(\tan\frac{\pi}{4} - \tan\frac{\pi}{6}\right)$

$= 2\left(1 - \frac{1}{\sqrt{3}}\right)$

(ii) $\log_a \frac{x^3}{\sqrt{y}} = \log_a x^3 - \log_a y^{\frac{1}{2}}$

$= 3\log_a x - \frac{1}{2}\log_a y$

$= 3(0.7) - \frac{1}{2}(0.4)$

$= 1.9$

e) $\int \frac{2}{2x+1}\,dx = \ln(2x+1) + c$

$y = \ln(2x+1) + c$

At $(1, \ln 3)$

$\ln 3 = \ln(2(1)+1) + c$

$= \ln(3) + c$

$\therefore c = 0$

$\therefore y = \ln(2x+1)$

f) (i) $f(x) = e^x - x$

$f'(x) = e^x - 1$

(ii) $f'(x) = 0$ $\qquad f''(x) = e^x$

$e^x - 1 = 0$ $\qquad > 0$

$e^x = 1$

$x = 0$ $\qquad \therefore$ Coordinates $(0, 1)$ is a local minimum.

$f(0) = e^0 - 0$

$= 1$

(ii) $\log_a \frac{x^3}{\sqrt{y}} = \log_a x^3 - \log_a y^{\frac{1}{2}}$

$= 3\log_a x - \frac{1}{2}\log_a y$

$= 3(0.7) - \frac{1}{2}(0.4)$

$= 1.9$

e) $\int \frac{2}{2x+1}\,dx = \ln(2x+1) + c$

$y_1 = \ln(2x_1+1) + c$

At $(1, \ln 3)$

$\ln 3 = \ln(2(1)+1) + c$

$= \ln(3) + c$

$\therefore c = 0$

$\therefore y = \ln(2x+1)$

f) (i) $f(x) = e^x - x$

$f'(x) = e^x - 1$

(ii) $f'(x) = 0$ $\qquad f''(x) = e^x$

$e^x - 1 = 0$ $\qquad > 0$

$e^x = 1$

$x = 0$ $\qquad \therefore$ Coordinates $(0, 1)$ is a local minimum.

$f(0) = e^0 - 0$

$= 1$

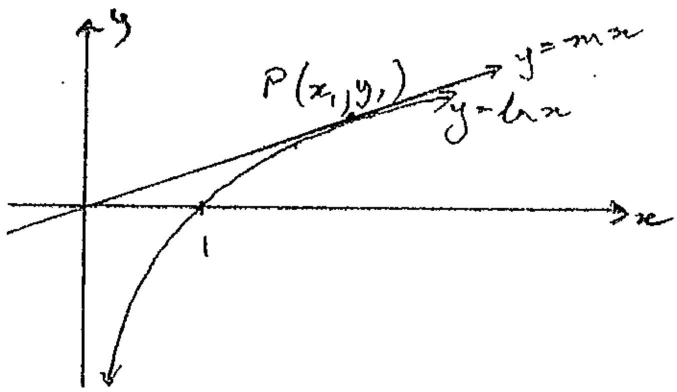
c) (i) $\int \frac{x}{x^2-3}\,dx = \frac{1}{2}\ln(x^2-3) + C$

(ii) $\int \frac{x^2-3}{x}\,dx = \int \frac{x^2}{x} - \frac{3}{x}\,dx$

$= \int x - \frac{3}{x}\,dx$

$= \frac{x^2}{2} - 3\ln x + C$

d) (i)



(ii) $y = \ln x$
$y' = \frac{1}{x}$

At x_1, $m = \frac{1}{x_1}$

(iii) At P,
$y_1 = mx_1$ and $y_1 = \ln x_1$

$\therefore mx_1 = \ln x_1$
$\frac{1}{x_1} x_1 = \ln x_1$
$1 = \ln x_1$
$\therefore x_1 = e$

Marker's Comments 2U 2015 Task 3

Q5 a) Do not approximate. Always give exact value unless asked to approximate.

b) iv) $\frac{d}{dx} \cos^2 3x$ caused difficulty.
Some brilliant students changed into $\cos 6x$.

d) Moving the graph up caused problems with students who could not put scale correctly on the x axis. (Only ½ mark penalty)

e) exact values required for final mark

f) Many many students found sector instead of segment.
Very generously marked with 1 mark for correct exact sector area.

HSC MATHEMATICS TASK 3 2015

MARKERS COMMENTS.

Question 6

No half marks awarded.

e) many students used $y - y_1 = m(x - x_1)$ to attempt to find the eqn of the tangent rather than the eqn of the curve

f) $A = \frac{1}{2} r^2 (\theta - \sin\theta)$

$= \frac{1}{2} (2\sqrt{3})^2 \left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

$= 6\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= 6\left(\frac{4\pi - 3\sqrt{3}}{6}\right)$

$= 4\pi - 3\sqrt{3}$ u^2

6. a) $x > 3$

b) (i) $\frac{d}{dx} \ln(2x+3) = \frac{2}{2x+3}$

(ii) $\frac{d}{dx} \ln(x-5)^7 = \frac{d}{dx} 7\ln(x-5)$

$= \frac{7}{x-5}$

c) $\int_0^3 x^2 - e^x \, dx = \left[\frac{x^3}{3} - e^x\right]_0^3$

$= \left(\frac{3^3}{3} - e^3\right) - \left(\frac{0^3}{3} - e^0\right)$

$= 9 - e^3 + 1$

$= 10 - e^3$

d) (i) $\log_5 1732 = \frac{\log_{10} 1732}{\log_{10} 5}$

$= 4.633\ldots$

$\doteqdot 4.63$ (3 sig-figs)

7. a) $V = \pi \int_0^{\frac{\pi}{4}} \tan^2 x \, dx$

$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x - 1 \, dx$

$= \pi \left[\tan x - x \right]_0^{\frac{\pi}{4}}$

$= \pi \left[\left(\tan\frac{\pi}{4} - \frac{\pi}{4} \right) - \left(\tan 0 - 0 \right) \right]$

$= \pi \left(1 - \frac{\pi}{4} \right)$

$= \pi - \frac{\pi^2}{4} \ u^3$

b) $2\cos^2 x + \sin x - 2 = 0$

$2(1 - \sin^2 x) + \sin x - 2 = 0$

$2 - 2\sin^2 x + \sin x - 2 = 0$

$2\sin^2 x - \sin x = 0$

$\sin x (2\sin x - 1) = 0$

$\sin x = 0$ for $0 \leq x \leq 2\pi$

$x = 0, \pi, 2\pi$

$2\sin x - 1 = 0$ for $0 \leq x \leq 2\pi$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \ \pi - \frac{\pi}{6}$

$\therefore \ x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi$

7. a) $V = \pi \int_0^{\frac{\pi}{4}} \tan^2 x \, dx$

$= \pi \int_0^{\frac{\pi}{4}} \sec^2 x - 1 \, dx$

$= \pi \left[\tan x - x \right]_0^{\frac{\pi}{4}}$

$= \pi \left[\left(\tan\frac{\pi}{4} - \frac{\pi}{4} \right) - \left(\tan 0 - 0 \right) \right]$

$= \pi \left(1 - \frac{\pi}{4} \right)$

$= \pi - \frac{\pi^2}{4} \ u^3$

b) $2\cos^2 x + \sin x - 2 = 0$

$2(1 - \sin^2 x) + \sin x - 2 = 0$

$2 - 2\sin^2 x + \sin x - 2 = 0$

$2\sin^2 x - \sin x = 0$

$\sin x (2\sin x - 1) = 0$

$\sin x = 0$ for $0 \leq x \leq 2\pi$

$x = 0, \pi, 2\pi$

$2\sin x - 1 = 0$ for $0 \leq x \leq 2\pi$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$\therefore x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi$

Markers Comments Q7

(a) Too many students didn't recognise the link of the Pythagorean Identity $1 + \tan^2 x = \sec^2 x$ (giving a standard integral).

Students need to be aware that

$$\frac{d}{dx} \cos^2 x \neq -\sin^2 x$$

as too many made this error thinking it was a log integral question.

(b) Again a weakness in identifying the identity $\cos^2 x = 1 - \sin^2 x$
Mistakes in recalling the identity correctly eg $\sin^2 x - 1$ was used

(c)(i) Well done

(ii) Splitting the fraction needs reinforcing as this was not well done

(d)(i) Common error was failing to have the line $y = mx$ passing through the Origin or showing it as a tangent to $y = \ln x$

(ii) good

(iii) Needed to use both $y = mx$ & $y = \ln x$ for substitution (solving simultaneously best approach)

(iv) good.