JAMES RUSE AGRICULTURAL HIGH SCHOOL

TERM TWO 2 UNIT ASSESSMENT TASK

1999

INSTRUCTIONS:

1. Time allowed is 85 minutes
2. This is an open book test
3. Show all necessary formulae and working
4. Marks may be deducted for untidy or careless work
5. Start each section on a new sheet of paper

Section A (12 marks)

1. A box contains 9 marbles of which 4 are red and 5 are blue. Two marbles are drawn at random from the box without replacement. Find the probability that they are the same colour.

2. If α and β are the roots of $x^2 + px + q = 0$ find:

   a) $\alpha + \beta$ and $\alpha\beta$

   b) $\alpha^2 + \beta^2$

   c) The equation whose roots are $\alpha - \beta$ and $\beta - \alpha$ in terms of p and q.

3. Solve $\dfrac{2-x}{x+4} - \dfrac{10(x+4)}{2-x} = 3$

SECTION B (12marks) Start a new page

1 Solve $x^2 + 4x > 5$ for all real values of x.

2.Scientists have found that $M = M_0e^{-kt}$, where M is in grams and t is in years, measures the mass of a element which is decaying at a rate proportional to the amount present. Measurements show that in 12 months, the mass of the element has fallen from 10g to 8.5g. How long will it take, to the nearest month, for it to reduce to 5g?

3.A particle has a constant acceleration of $3m/s^2$, if it starts from rest at the origin, find an expression for the displacement, x, in terms of t.

4.Ben buys three tickets in a school raffle, Daniel buys four tickets in the same raffle. There are 200 tickets sold altogether and three prizes with third being drawn first, then second and then first. Find the probability that:

a) Ben wins third prize.
b) Daniel wins second prize.
c) Ben wins one prize, correct to three decimal places; show all working.

SECTION C (12 marks) Start a new page

1.Jennifer throws three dice and observes the number on the top face of each die. Find the probability that:

a) no two die show the same number
b) all the dice show the same number
c) only two of them show the same number

2.Find the exact values of p for which the following equation has equal roots:

$3x^2 - (p-4)x - (2p - 1) = 0$

3.A radioactive substance disintegrates proportionally to the amount present at the given time according to the formula $R(t) = R(0)e^{-kt}$, where k is a constant and t is in years.

a) At what time will there be exactly half the original amount left? Express your answer in terms of k.
b) It is known that radioactive kryptonite has a half-life of 5600 years. Some decomposed kryptonite was found at a mining site and an analysis showed that one fifth of the original amount has decomposed. How long has the kryptonite been at the mining site to the nearest 10 years ?

SECTION D (12 marks) Start a new page

1. The displacement x metres at time t seconds of a particle moving in a straight line is given by $x = 1 + 3\cos^2 t$. Find:
   a) an expression for the velocity in terms of t.
   b) an expression for the acceleration in terms of t.
   c) hence or otherwise show that $a^2 = 36 - 4v^2$

2. For $P=P_0e^{kt}$ where P is the population of an urban area and P_0 is the population in 1980, if in the beginning of 1980 the population was 15 000 and at the end of 1990 it was 24 000, find when the population will first exceed 30 000, (to the nearest month).

3. a) What is the probability of obtaining at least one six when a die is rolled three times?

   b) Find the minimum number of throws of a die needed to ensure a 60% chance of obtaining at least one six.

SECTION E (12 marks) Start a new page

1. The velocity of a particle moving in a straight line is given by $v = e^{-2t}\sin t$ where t is in seconds and the displacement in metres. Initially the particle is at rest and $0 \le t \le \pi$.

a) When does the particle next come to rest ?
b) Find the smallest positive value of t for which the acceleration of the particle is zero (answer to be given correct to two decimal places).
c) Find the maximum velocity of the particle correct to one decimal place.

2. Show that the line $y = mx + 5$ is a tangent to the curve $y = x^2 - 3x + 6$ when $m = -1$ or -5.

3. A number is chosen at random from the numbers 200 to 999 inclusive. Find the probability that two and only two consecutive digits are the same.

THE END

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \ne -1; \quad x \ne 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \ne 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \ne 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \ne 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \ne 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \ne 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \ne 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x>0$

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SECTION A (12 marks)

1. P(same colour) = RR or BB

(1) $= \frac{4}{9} \times \frac{3}{8} + \frac{5}{9} \times \frac{4}{8}$

$= \frac{4}{9}$ (1)

2. a) $\alpha + \beta = -p$

$\alpha\beta = q$ (1)

b) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$ (1)

$= p^2 - 2q$ (1)

c) $\alpha - \beta$, $\beta - \alpha$ are the roots

sum of roots $= 0$ (1)

product of roots $= (\alpha-\beta)(\beta-\alpha)$

$= 2\alpha\beta - 1(\alpha^2+\beta^2)$

$= 2q - (p^2 - 2q)$

$= 4q - p^2$ (1)

new equation is:

$x^2 + 4q - p^2 = 0$ (1)

let $y = \frac{2-x}{x+4}$

$y - \frac{10}{y} = 3$ (1)

$y^2 - 3y - 10 = 0$

$(y-5)(y+2) = 0$

$\therefore y = 5$ or -2 (1)

$\frac{2-x}{x+4} = 5$ $\qquad$ $\frac{2-x}{x+4} = -2$

$x = 5x + 20$ $\qquad$ $2 - x = -2x - 8$

$6x = -18$

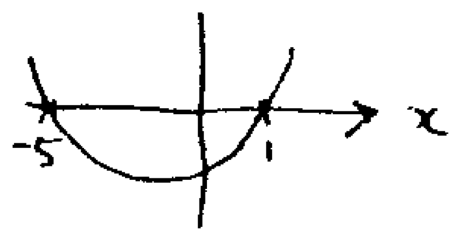
$\therefore x = -3$ (1) $\qquad$ $x = -10$ (1)

SECTION B (12 marks)

Q1. $x^2 + 4x > 5$

$x^2 + 4x - 5 > 0$

$(x+5)(x-1) > 0$ (1)



$x > 1$ and $x \le -5$ (1)

Q2 $M = M_0 e^{kt}$

$8.5 = 10e^k$

$\ln(0.85) = k$ (1)

$\therefore \frac{1}{2} = e^{\ln(0.85)t}$

$t = \frac{\ln(0.5)}{\ln(0.85)}$ (1)

$= 4.265$

$= 4$yrs 3months (1)
(nearest month)

Q3 $a = 2$

$v = 3t + c$

but $t = 0, v = 0$ $\therefore c = 0$ (1)

$\therefore v = 3t$

$x = \frac{3t^2}{2} + k$

but $t = 0, x = 0, k = 0$

$\therefore x = \frac{3t^2}{2}$ (1)

4. a) P(3rd prize) $= \frac{3}{200}$ (1)

b) P(Daniel wins second prize) (1)

$= \frac{196}{200} \times \frac{4}{199} \times \frac{195}{195}$ (1)

$= 0.0197$

c) P(Ben's 3rd, 2nd or 1st prize)

$= 3 \times \left(\frac{3}{200} \times \frac{197}{199} \times \frac{196}{198}\right)$ (1)

$= 0.044097761$ (1)

$\doteqdot 0.044$ (correct to 3 dec. pl) (1)

SECTION C (12 marks)

1. a) P(all different) $= 1 \times \frac{5}{6} \times \frac{4}{6}$

$= \frac{5}{9}$ (1)

b) P(all same) $= 1 \times \frac{1}{6} \times \frac{1}{6}$

$= \frac{1}{36}$ (1)

c) P(2 are same) $= 3 \times \left(1 \times \frac{1}{6} \times \frac{5}{6}\right)$

$= \frac{5}{18}$ (1)

2. $b^2 - 4ac = 0$ for equal roots

$(p-4)^2 + 4(3)(2p-1) = 0$

$p^2 - 8p + 16 + 24p - 12 = 0$

$p^2 + 16p + 4 = 0$ (1)

$p = \frac{-16 \pm \sqrt{256-16}}{2}$ (1)

$= \frac{-16 \pm 4\sqrt{15}}{2}$

$\therefore p = -8 \pm 2\sqrt{15}$ (1)

3. a) $\frac{1}{2}R(0) = R(0)e^{-kt}$

$\frac{1}{2} = e^{-kt}$ (1)

$\ln(\frac{1}{2}) = -kt$

$\therefore t = \frac{\ln 2}{k}$ (1)

b) $5600 = \frac{\ln 2}{k}$

$\therefore k = 1.23776 \times 10^{-4}$ (1)

$\frac{1}{5} = e^{-\frac{\ln 2}{5600}t}$ (1)

$t = \frac{5600\ln(1/5)}{-\ln 2}$ (1)

$= 13002.79733$

$\therefore t \doteqdot 13000$ yrs (1)

SECTION D (12 marks)

Q1. a) $v = -6\sin t . \cos t$

b) $a = -6\sin t(-\sin t) + \cos t.$

$= 6\sin^2 t - 6\cos^2 t$

$= 6(\sin^2 t - \cos^2 t)$

c) $a^2 = 36(\sin^2 t - \cos^2 t)^2$

(1) $= 36(\sin^4 t + \cos^4 t - 2\sin^2 t \cos^2 t)$

$= 36\left[(\sin^2 t + \cos^2 t)^2 - 2\sin^2 t \cos^2 t - 2\sin^2 t \cos^2 t\right]$

(1) $= 36\left[1 - 4\sin^2 t \cos^2 t\right]$

(1) $= 36\left[1 - \frac{1}{9}v^2\right]$ (as $v^2 = 36\ldots$)

$= 36 - 4v^2$

QED.

Q2. $P = P_0 e^{kt}$

$P_0 = 15000$ (1980)

when $t = 11$, $P = 24000$

$\therefore 24000 = 15000e^{11k}$

(1) $\ln(8/5) = 11k$

$\therefore k = 0.0427276$

$30000 = 15000e^{0.0427276t}$

$\ln(2) \ldots$

$= 16.222468321$

$\therefore$ In March 1995, the population will exceed 30 000.

a) $P(\text{at least } 1\ 6) = 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}$ (1)

b) $P(\text{at least one six}) = 1 - P(\text{no sixes})$

Prob. $= 1 - \left(\frac{5}{6}\right)^n$

$1 - \left(\frac{5}{6}\right)^n = \frac{3}{5}$ (1)

$1 - \frac{3}{5} = \left(\frac{5}{6}\right)^n$

$\frac{2}{5} = \left(\frac{5}{6}\right)^n$

$\log_{(5/6)}(2/5) = n$ (1)

$\therefore n = \frac{\ln(2/5)}{\ln(5/6)}$

$= 5.025685$

$\therefore$ Must be 6 throws. (1)

SECTION E (12 marks)

$v = e^{-2t}.\sin t$

$a = \sin t.(-2e^{-2t}) + e^{-2t}(\cos t)$

$= e^{-2t}(-2\sin t + \cos t)$

$a = 0 \quad \therefore e^{-2t}(\cos t - 2\sin t) = 0$

$e^{-2t} \neq 0 \qquad \cos t = 2\sin t$ (1)

$\frac{1}{2} = \tan t$

$t = 0.463648^c$

(1) $t = 0.46$ seconds

ie $0 = e^{-2t}\sin t$ (1)

$e^{-2t} \neq 0 \quad \sin t = 0$ when (1)

$t = 0, \pi, 2\pi \ldots$

$\therefore$ next comes to rest when $t = \pi$

c) max. velocity when $a = 0$,

$\therefore t = 0.46 \quad \therefore v = e^{-2(0.46)}.\sin 0.46$ (1)

$= 0.178$

Q2. $mx + 5 = x^2 - 3x + 6$

$0 = x^2 - 3x - mx + 6 - 5$

$0 = x^2 - x(3 + m) + 1$

$m = -1, \quad \Delta = 4 - 4 = 0$ (1)

$\therefore$ line is a tangent

$m = -5, \quad \Delta = 4 - 4 = 0$ (1)

$\therefore$ line is a tangent

$\therefore$ when $m = -1$ or -5, $y = mx + 5$ is a tangent to $y = x^2 - 3x + 6$

Q3. now in the 200's we have

200	220
211	221
233	222
244	⋮
255	
266	
277	229
288	ie 10 numbers (1)
299	
ie 9 numbers	

there are 8 possibilities

$\therefore$ no. of events $= 9 \times 8 + 8 \times 10$

$= 72 + 80$

$= 152$ (1)

Sample space $= 999 - 199 = 800$

$\therefore$ prob. $= \frac{152}{800} = \frac{19}{100}$ (1)