Year 12 MATHEMATICS
TERM 2 OPEN BOOK ASSESSMENT TASK 2001

ne Allowed: 85 Minutes

- This is an open book exam
- Show all necessary working
- Marks may be deducted for poorly set out work
- Start each section on a new page

CTION A (12 Marks) START A NEW PAGE

Marks

Let the roots of the equation $2x^2 - 3x + 8 = 0$ be α and β. Evaluate $\alpha\beta^2 + \alpha^2\beta$. 2

30 blue tickets and 20 red tickets are sold in a raffle in which there are three prizes. A ticket is drawn at random and then discarded after each draw. First prize is drawn first, then the second prize and then the third prize.

(i) What is the probability that all the prizes are won by red tickets? 1

(ii) What is the probability that at most one blue ticket wins a prize? 2

A body moving in a straight line so that its position from the origin (x metres) after t seconds is given by $x = \dfrac{50(t-3)}{e^t}$.

(i) Find its velocity after t seconds. 2

(ii) Find its initial position. 1

(iii) Find the greatest positive displacement. 3

(iv) Find the body's maximum speed. 1

CTION B (12 Marks) START A NEW PAGE

Find p if the roots of $9x^2 - 3x + p = 0$ are equal. 2

The rate of increase of the population P(t) of a certain town is governed by the equation $\dfrac{dP}{dt} = kP$ where k is a constant and t is the time in years. If the population doubles at the end of 15 years,

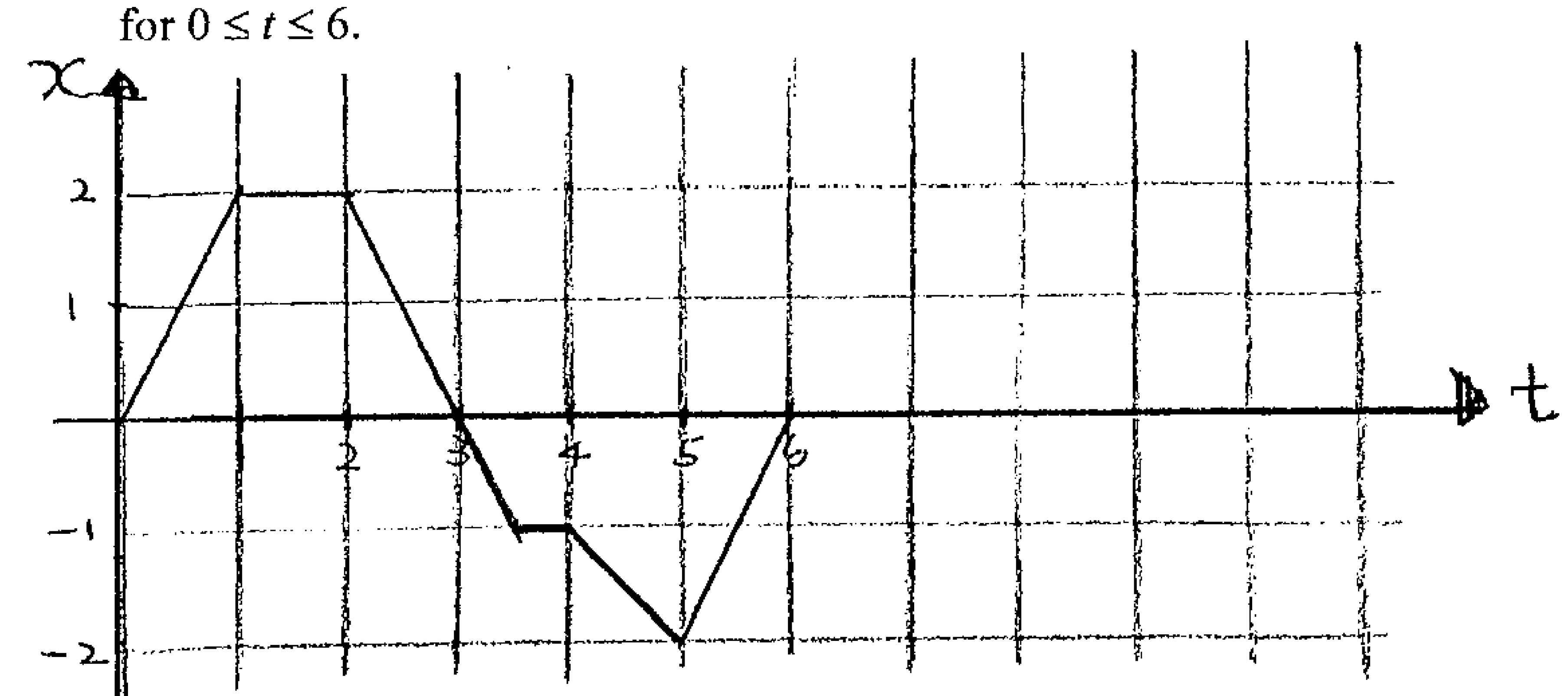
(i) Find the exact value of k. 2

(ii) In which year will the town reach a population 3 times that it had at the beginning of 2001? 2

(iii) Given that at the beginning of 2001, the population was 18.9 million, what will the population be at the beginning of 2011? (Answer to the nearest thousand) 2

The probability that the Olympia soccer team will win its game against the Macaroni soccer team in any particular match is 0.6. If the two teams meet on 3 occasions what is the probability, as a fraction, that the Macaroni team will win at least one of their games. 2

4. The graph shows the displacement (in metres) of a particle at any time t seconds, for $0 \le t \le 6$.

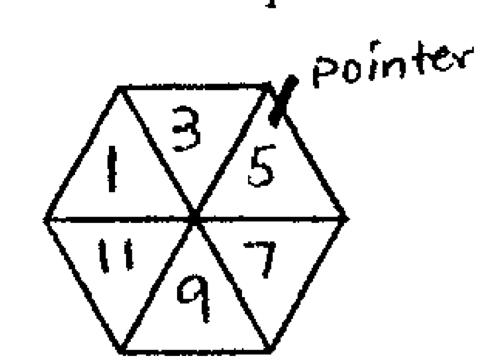


(i) Between which times is the particle at rest? 1

(ii) What is the distance travelled in the 5th second ? 1

SECTION C (12 Marks) START A NEW PAGE

1. A hexagon has 6 equal sections marked with the numbers **1, 3, 5, 7, 9, 11**. When the hexagon is spun, each section has an equally likely chance of stopping at the pointer. The hexagon is spun twice and the product of the two numbers is recorded.



(i) Draw a dot / lattice diagram to represent these outcomes. 1

(ii) Find the probability that the product is an even number. 1

(iii) Find the probability that the product is a number larger that 21 but smaller than 45. 1

(iv) Find the probability that the product is a multiple of 5, if it is known that the first number is a 3. 1

2. a) Rewrite $y = 3x^2 + 6x - 7$ in the form of $y = a(x+b)^2 + c$, and hence or otherwise find the minimum value of the function. 3

b) Find the values of k such that $k(k+3) + (k+3)x - x^2$ is negative definite. 3

3. The half-life of U-235 (an isotope of uranium) is 7×10^8 years. Find the growth constant. 2

SECTION D (12 Marks) START A NEW PAGE

1. The roots of the quadratic equation $kx^2 - 2x + h = 0$ have a sum of $\dfrac{3}{2}$ and a product of 2. Find the values of k and h. 4

2. The acceleration in metres per second per second of a moving object is given at time t seconds by $a = 2\pi^2 \cos\left(\frac{\pi t}{2}\right)$. Initially, the object is at the origin and travelling with velocity 2π m/s.

(i) Find the velocity (v) and the displacement (x) as a function of t. 4
(ii) Find for t in the range $0 \le t \le 4$, the values of t for which the object is stationary. 3
(iii) What is its displacement when it first comes to rest. 1

SECTION E (12 Marks) START A NEW PAGE

1. A particle is initially at rest at the origin. It's acceleration is a linear function of time t seconds. Initially, the acceleration is 16 m/s^2 and decreases to zero after 4 seconds.

(i) Find the equations for acceleration, velocity and displacement for this particle in terms of t. 4
(ii) Calculate the velocity and the position after 4 seconds. 1

2. A particle moving in a straight line was initially at the origin. The velocity of the particle at time t is given by $v = v_o e^{-\lambda t}$ where v_o is the initial velocity of the particle, regarded as positive, and λ is a positive constant.

(i) Prove that when $t = \frac{1}{\lambda}\log_e 2$ the magnitude of the acceleration will be half its initial value. 4

(ii) Show that the displacement of the particle from the origin is always less than $\frac{v_o}{\lambda}$ 3

~ END OF EXAM ~

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0.$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0.$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0.$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0.$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0.$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0.$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0.$$

$$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a.$$

$$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \ln\left\{x + \sqrt{(x^2-a^2)}\right\},\ |x| > |a|$$

$$\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx = \ln\left\{x + \sqrt{(x^2+a^2)}\right\}$$

NOTE: $\ln x = \log_e x,\ x > 0$

SECTION A (12 MARKS)

$2x^2 - 3x + 8 = 0$

$\alpha + \beta = \frac{3}{2}$
$\alpha\beta = 4$ $\Big\rangle$ $\boxed{1 \text{ mark}}$

$\therefore \alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha+\beta)$
$= 6$ $\boxed{1\text{mark}}$

(i) $P(RRR) = \frac{20}{50} \times \frac{19}{49} \times \frac{18}{48}$
$= \frac{57}{980}$ $\boxed{1 \text{ mark}}$

(ii) $P(\text{no blue or } 1B) = P(RRR) + 3P(BRR)$
$= \frac{57}{980} + 3\left(\frac{30}{50} \times \frac{20}{49} \times \frac{19}{48}\right)$
$\doteq \frac{171}{490}$ $\boxed{2 \text{ marks}}$

$x = \frac{50(t-3)}{e^t}$

(i) Use Quotient Rule or Product Rule
$v = \frac{dx}{dt}$
$= \frac{50(4-t)}{e^t}$ $\boxed{2 \text{ marks}}$

(ii) Initial Position when $t=0$
$\therefore x = \frac{50(-3)}{1}$
$= -150$ $\boxed{1\text{mark}}$

(iii) when $v=0$, $t=4$
$\therefore x = \frac{50}{e^4}$ $\boxed{1 \text{ mark}}$

as $t \to \infty$ $x \to 0$ $\boxed{1 \text{ mark}}$

$\therefore$ max positive displacement is
$x = \frac{50}{e^4}$ $\boxed{1 \text{ mark}}$

(iv) as $t \to \infty$, $v \to 0$
$\therefore$ max. speed occurs when $t=0$.
$\therefore v = \frac{50(4)}{e^0} = 200 \text{ m/s}$ $\boxed{2 \text{ marks}}$

SECTION B (12 marks)

1. $9x^2 - 3x + p = 0$
Equal roots when $\Delta = 0$
$\therefore 9 - 4(9)(p) = 0$
$9 - 36p = 0$
$p = \frac{1}{4}$ $\boxed{2 \text{ marks}}$

2. (i) $P = P_0 e^{kt}$
$t = 15$ $P = 2P_0$
$\therefore 2 = e^{15k}$
$\ln 2 = 15k$
$\therefore k = \frac{\ln 2}{15}$ $\boxed{2 \text{ marks}}$

(ii) $P = 3P_0$ find t.
$\therefore 3 = e^{\frac{\ln 2}{15}t}$
$\therefore \ln 3 = \frac{\ln 2(t)}{15}$
$\frac{15 \ln 3}{\ln 2} = t$
$\therefore t \approx 23.77$
$\therefore$ year 2024 $\boxed{2 \text{ marks}}$

(iii) $t = 10$ $P_0 = 18.9 \times 10^6$
$P = 18.9 \times 10^6 e^{\frac{\ln 2}{15}(10)}$ $\boxed{2 \text{ marks}}$
$\doteq 30\ 001\ 879.88$
$= 30\ 002\ 000$ (to nearest thousand)

3. $P(\text{at least 1 win})$
$= 1 - P(\text{no win})$
$= 1 - (0.4)^3$
$= 1 - \left(\frac{4}{10}\right)^3$
$= 1 - \frac{64}{1000}$ $\boxed{2 \text{ marks}}$
$= \frac{936}{1000}$
$= \frac{117}{125}$

Section B (continued)

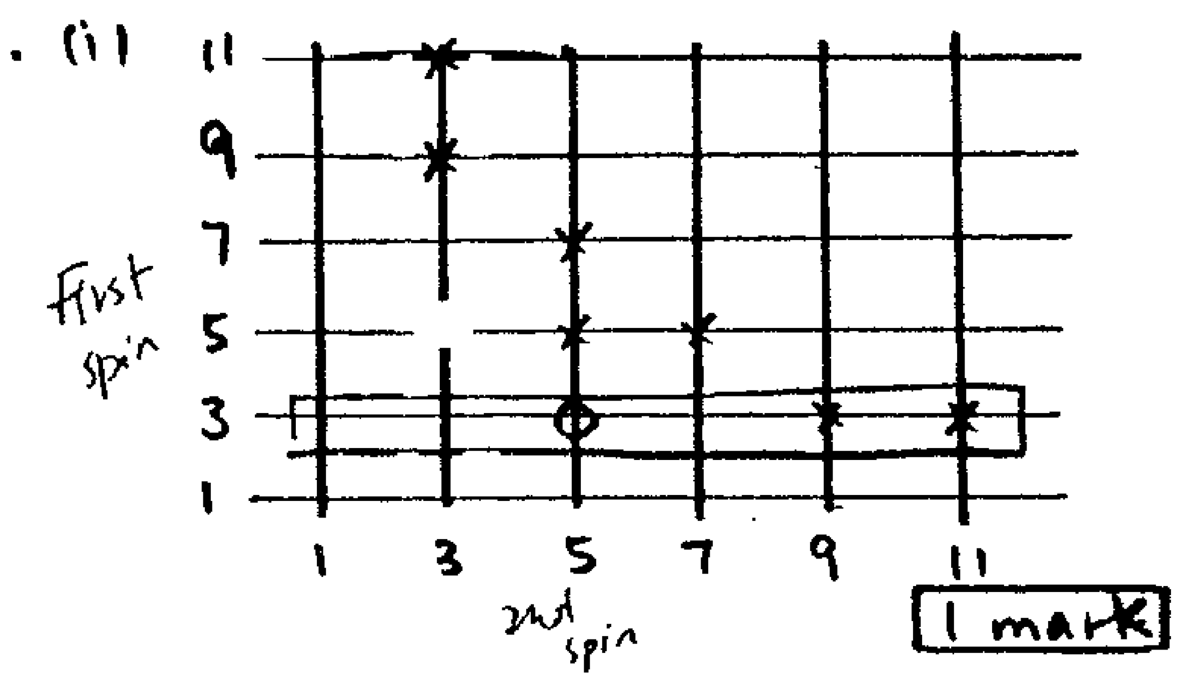
4. (i) particle at rest when
$1 \le t \le 2$ and $3\frac{1}{2} \le t \le 4$

(ii) distance is 1 metre $\boxed{2 \text{ marks}}$

3. $A = A_0 e^{-kt}$
$\frac{1}{2}U_0 = U_0 e^{-k \times 7 \times 10^8}$
$\frac{1}{2} = e^{-k \times 7 \times 10^8}$
$-\ln 2 = -k \times 7 \times 10^8$
$\therefore k = \frac{\ln 2}{7 \times 10^8}$

SECTION C (12 MARKS)

1. (i)



$\boxed{1 \text{ mark}}$

(ii) $P(\text{even}) = 0$ $\boxed{1\text{mark}}$

(iii) $P(21 < \text{number} < 45)$
$= \frac{7}{36}$ $\boxed{1 \text{ mark}}$

(iv) $P(\text{mult. } 5 \text{ given } = 3)$
$= \frac{1}{6}$ $\boxed{1 \text{ mark}}$

2. a) $y = 3x^2 + 6x - 7$
$y = 3(x^2 + 2x) - 7$
$= 3(x^2 + 2x + 1) - 7 - 3$
$= 3(x+1)^2 - 10$ $\boxed{2 \text{ marks}}$

$\therefore$ min. value is -10 $\boxed{1\text{mark}}$

b) $k(k+3) + (k+3)x - x^2$
negative definite when $a < 0$ and $\Delta < 0$
$\therefore (k+3)^2 - 4(-1)(k+3)k < 0$
$(k+3)(k+3 + 4k) < 0$
$(k+3)(5k+3) < 0$
$\therefore -3 < k < -\frac{3}{5}$ $\boxed{3 \text{ marks}}$

SECTION D (12 marks)

① $kx^2 - 2x + h = 0$
Given $\alpha + \beta = \frac{3}{2}$
$\alpha\beta = 2$
$\therefore \frac{3}{2} = \frac{2}{k}$ and $\frac{h}{k} = 2$
$\therefore k = \frac{4}{3}$
$\therefore h = 2k$
$\therefore h = \frac{8}{3}$ $\boxed{4 \text{ marks}}$

1. $a = 2\pi^2 \cos\left(\frac{\pi t}{2}\right)$
$t = 0$, $x = 0$, $v = 2\pi$

(i) $v = \int 2\pi^2 \cos\left(\frac{\pi t}{2}\right) dt$
$= \frac{2\pi^2 \times \sin\left(\frac{\pi t}{2}\right)}{\frac{\pi}{2}} + c$
$= 4\pi \sin\left(\frac{\pi t}{2}\right) + c$ $\boxed{2 \text{ mark}}$

when $t = 0$, $v = 2\pi$
$\therefore 2\pi = 4\pi(\sin(0)) + c$
$\therefore c = 2\pi$
$\therefore v = 4\pi \sin\left(\frac{\pi t}{2}\right) + 2\pi$

$x = \int v \, dt$
$= \frac{-4\pi \cos\left(\frac{\pi t}{2}\right)}{\frac{\pi}{2}} + 2\pi t + c$
$= -8 \cos\left(\frac{\pi t}{2}\right) + 2\pi t + c$
when $t = 0$, $x = 0$
$\therefore 0 = -8\cos(0) + 0 + c$
$\therefore c = 8$
$\therefore x = -8\cos\left(\frac{\pi t}{2}\right) + 2\pi t + 8$ $\boxed{2 \text{ mark}}$

SECTION D continued

i) $v=0$ find t for $0 \le t \le 4$

$0 = 4\pi \sin\left(\frac{\pi t}{2}\right) + 2\pi$

$\sin\left(\frac{\pi t}{2}\right) = -\frac{1}{2}$ (t in 3rd + 4th quadrants)

$\therefore \frac{\pi t}{2} = \pi + \frac{\pi}{6},\ 2\pi - \frac{\pi}{6},\ 3\pi + \frac{\pi}{6}, \ldots$

$t = \frac{7}{3}, \frac{11}{3}, \frac{19}{3} \ldots$

but $0 \le t \le 4$

$\therefore$ $\underline{t = \frac{7}{3} \text{ and } \frac{11}{3}}$ only

$\boxed{3 \text{ marks}}$

iii) when $t = \frac{7}{3}$

$x = -8\cos\left(\frac{\pi}{2} \times \frac{7}{3}\right) + 2\pi\left(\frac{7}{3}\right) + 8$

$= 4\sqrt{3} + \frac{14\pi}{3} + 8$

$\doteqdot \underline{29.6}$ metres

$\boxed{1 \text{ mark}}$

SECTION E (12 Marks)

1.(i) $\ddot{x} = at + b$

$t=0 \quad \ddot{x} = 16 \Rightarrow \underline{b = 16}$

$t=4 \quad \ddot{x} = 0$

$\therefore 0 = 4a + 16$

$\therefore \underline{a = -4}$

$\therefore \underline{\ddot{x} = -4t + 16}$

$\boxed{2 \text{ marks}}$

$\dot{x} = -2t^2 + 16t + c$

when $t=0 \quad \dot{x} = 0 \Rightarrow c = 0$

$\therefore \underline{\dot{x} = -2t^2 + 16t}$

$\boxed{1 \text{ mark}}$

$x = \frac{-2t^3}{3} + 8t^2 + c$

when $t=0,\ x=0 \Rightarrow c=0$

$\therefore \underline{x = \frac{-2t^3}{3} + 8t^2}$

$\boxed{1 \text{ mark}}$

1(ii) when $t=4$

$v = 32$ m/s

$x = \underline{85\frac{1}{3}}$ metres.

$\boxed{1 \text{ mark}}$

2. (i) $v = v_0 e^{-\lambda t}$

$a = -\lambda v_0 e^{-\lambda t}$

$\boxed{1 \text{ mark}}$

when $t=0$, $v = v_0$

$\therefore \underline{a = -\lambda v_0}$

$\boxed{1 \text{ mark}}$

when $a = \frac{1}{2} - \lambda v_0$ we get

$-\frac{1}{2}\lambda v_0 = -\lambda v_0 e^{-\lambda t}$

$\therefore \frac{1}{2} = e^{-\lambda t}$

$\ln\frac{1}{2} = -\lambda t$

$\ln 1 - \ln 2 = -\lambda t$

$-\ln 2 = -\lambda t$

$\therefore \underline{t = \frac{\ln 2}{\lambda}}$

$\boxed{2 \text{ marks}}$

(ii) $x = \int v\, dt$

$= \frac{v_0 e^{-\lambda t}}{-\lambda} + c$

when $t=0$, $x=0$

$\therefore 0 = \frac{v_0 e^0}{-\lambda} + c$

$\therefore c = \frac{v_0}{\lambda}$

$\therefore x = \frac{v_0 e^{-\lambda t}}{-\lambda} + \frac{v_0}{\lambda}$

$\underline{x = \frac{v_0}{\lambda}\left(1 - e^{-\lambda t}\right)}$

$\boxed{2 \text{ marks}}$

as $t \to \infty$, $e^{-\lambda t} \to 0$

$\therefore x \to \frac{v_0}{\lambda}$

$\underline{\underline{0 \le x < \frac{v_0}{\lambda}}}$

$\boxed{1 \text{ mark}}$

— END OF TEST —