NORTH SYDNEY BOYS' HIGH SCHOOL
2008 HSC Course Assessment Task 3

# MATHEMATICS (ADVANCED)

**General instructions**

- Working time – 60 minutes.
- Write in the booklet provided.
- Each new question is to be started on a new booklet.
- Write using blue or black pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets within this paper and hand to examination supervisors.
- Unless otherwise specified, log means logarithm to base $e$.

**Class teacher** (please ✔)

- ◯ Mr Lam
- ◯ Mr Taylor
- ◯ Mr Fletcher
- ◯ Mr Lowe
- ◯ Mr Ireland

**STUDENT NUMBER:** ..................... **PAGES USED**: .............................

Marker's use only.

| PART | A | B | C | D | E | Total | % |
|---|---|---|---|---|---|---|---|
| MARKS | $\overline{\phantom{00}}$ 15 | $\overline{\phantom{00}}$ 12 | $\overline{\phantom{00}}$ 14 | $\overline{\phantom{00}}$ 10 | $\overline{\phantom{00}}$ 11 | $\overline{\phantom{00}}$ 62 | |

**Part A** (15 Marks) Commence a **new** sheet. **Marks**

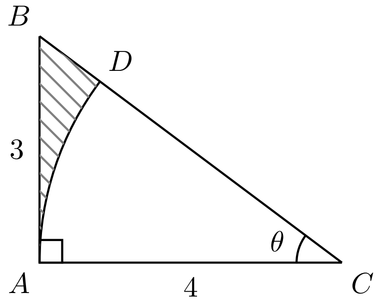
**1** Write correct to 3 decimal places:

(a) $45^\circ$ to radians. **2**

(b) $\cos 2.9$. **2**

**2** What is the period and amplitude of the curve $y = 3\cos 2x$? **4**
Sketch the curve for $0 \le x \le 2\pi$.

**3** From a right angled triangle $\triangle ABC$ with the right angle at $A$, an arc is drawn from $A$ with centre $C$ and radius $AC = 4$ cm. This arc meets the hypotenuse at $D$. This is shown in the diagram below.




(a) Find $\angle ACB$ in radians correct to 2 decimal places. **2**

(b) Find the length of the arc $AD$ correct to 2 decimal places. **2**

(c) Find the area bounded by the region $AB$, $BD$ and arc $AD$ correct to 2 decimal places. **3**

**Part B** (12 Marks) Commence a **new** sheet.

**1** Differentiate:

(a) $\dfrac{\tan x}{2x+1}$. **2**

(b) $\sin^3 x$. **2**

(c) $x^3 e^{-3x}$. **2**

(d) $\log_e \left(\dfrac{2x+1}{3x-7}\right)$. **3**

**2** Find the equation of the normal to the curve $y = x \sin x$ where $x = \dfrac{\pi}{2}$. **3**

**Part C** (14 Marks) Commence a **new** sheet. **Marks**

**1** Sketch $y = \log_{10}(x-2)$, showing essential features. State its domain and range. **3**

**2** Find the primitive of:

(a) $\dfrac{2x}{x^2+1}$. **2**

(b) $\dfrac{e^{2x}}{e^{2x}+4}$. **2**

(c) $\dfrac{2}{x} + 5e^x$. **2**

**3** The minute hand of a town clock is 1.75 m long.

(a) How far does the tip move in 35 min? (Answer correct to 1 d.p.) **3**

(b) How long does it take the hand to rotate through $\dfrac{\pi}{15}$ radians? **2**

**Part D** (10 Marks) Commence a **new** sheet.

**1** Find the *exact* area of a minor segment of a circle, radius $5\sqrt{2}$ cm, cut off by a 10 cm chord. **3**

**2** (a) Sketch the curve $f(x) = 2e^{-x}$, clearly showing the $y$ intercept. Using this, draw $y = -f(x)$. **2**

(b) $y = f(-x)$. **1**

**3** Evaluate:

(a) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \cos 2x \, dx$ **2**

(b) $\displaystyle\int_{0}^{\frac{\pi}{4}} \frac{1}{2}x - \sin(2x) \, dx$. **2**

**Part E** (11 Marks) Commence a **new** sheet.

**1** (a) Show that $\dfrac{d}{dx}\left(x \ln x - x\right) = \ln x$. **2**

(b) Hence or otherwise, find **2**

$$\int \ln\left(x^2\right) \, dx$$

**2** (a) What are the coordinates and the nature of the stationary point for the curve $y = \dfrac{e^x}{x^2+1}$? **4**

(b) What is the range of this function? Sketch this curve, showing any intercepts. **3**

**End of paper.**

**STANDARD INTEGRALS**

| | | |
|---|---|---|
| $\int x^n \, dx$ | $= \frac{1}{n+1} x^{n+1} + C,$ | $n \neq -1; \quad x \neq 0$ if $n < 0$ |
| $\int \frac{1}{x} \, dx$ | $= \ln x + C,$ | $x > 0$ |
| $\int e^{ax} \, dx$ | $= \frac{1}{a} e^{ax} + C,$ | $a \neq 0$ |
| $\int \cos ax \, dx$ | $= \frac{1}{a} \sin ax + C,$ | $a \neq 0$ |
| $\int \sin ax \, dx$ | $= -\frac{1}{a} \cos ax + C,$ | $a \neq 0$ |
| $\int \sec^2 ax \, dx$ | $= \frac{1}{a} \tan ax + C,$ | $a \neq 0$ |
| $\int \sec ax \tan ax \, dx$ | $= \frac{1}{a} \sec ax + C,$ | $a \neq 0$ |
| $\int \frac{1}{a^2 + x^2} \, dx$ | $= \frac{1}{a} \tan^{-1} \frac{x}{a} + C,$ | $a \neq 0$ |
| $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$ | $= \sin^{-1} \frac{x}{a} + C,$ | $a > 0, -a < x < a$ |
| $\int \frac{1}{\sqrt{x^2 - a^2}} \, dx$ | $= \ln \left( x + \sqrt{x^2 - a^2} \right) + C,$ | $x > a > 0$ |
| $\int \frac{1}{\sqrt{x^2 + a^2}} \, dx$ | $= \ln \left( x + \sqrt{x^2 + a^2} \right) + C$ | |

NOTE: $\ln x = \log_e x, \; x > 0$

# Solutions

## Part A

**1** (a) (2 marks)

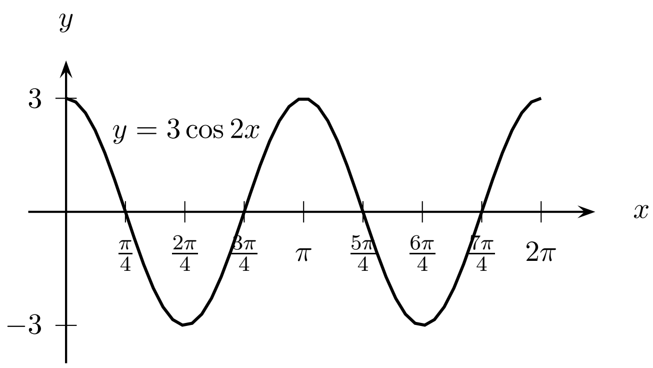
$$\frac{43^\circ \times \pi}{180^\circ} = 0.750 \text{ (3 d.p.)}$$

(b) (2 marks)

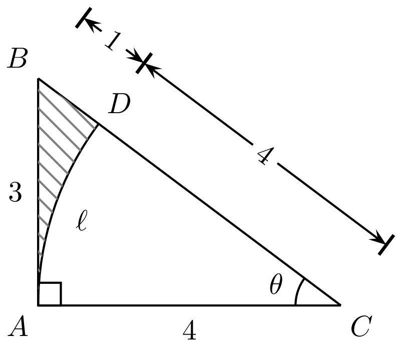
$$\cos 2.9 = -0.971 \text{ (3 d.p.)}$$

**2** (4 marks)

- $T = \frac{2\pi}{2} = \pi$
- $a = 3$




**3** (a) (2 marks)




$$\tan\theta = \tfrac{3}{4}$$
$$\theta = \tan^{-1}\tfrac{3}{4} = 0.64$$

(b) (2 marks)

$$\begin{aligned}\ell = r\theta &= 4 \times \tan^{-1}\tfrac{3}{4} = 2.57\\ &= 2.56 \text{ (if using 0.64 as } \theta)\end{aligned}$$

(c) (3 marks)

$$\begin{aligned}A_{ABD} &= A_\triangle - A_{\text{sect}}\\ &= \left(\tfrac{1}{2}bh\right) - \left(\tfrac{1}{2}r^2\theta\right)\\ &= \left(\tfrac{1}{2}\times 3 \times 4\right) - \left(\tfrac{1}{2} \times 4^2 \times \tan^{-1}\tfrac{3}{4}\right)\\ &= 6 - 5.148\cdots = 0.852\end{aligned}$$

## Part B

**1** (a) (2 marks)

$$y = \frac{\tan x}{2x+1}$$

$$u = \tan x \qquad v = 2x+1$$
$$u' = \sec^2 x \quad v' = 2$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{vu' - uv'}{v^2}\\ &= \frac{(2x+1)\sec^2 x - 2\tan x}{(2x+1)^2}\end{aligned}$$

(b) (2 marks)

$$y = \sin^3 x = (\sin x)^3$$

$$y(u) = u^3 \qquad u(x) = \sin x$$
$$y'(u) = 3u^2 \quad u'(x) = \cos x$$

$$\begin{aligned}y'(x) &= y'(u) \times u'(x)\\ &= 3\sin^2 x\cos x\end{aligned}$$

(c) (2 marks)

$$y = x^3 e^{-3x}$$

$$u = x^3 \qquad v = e^{-3x}$$
$$u' = 3x^2 \quad v' = -3e^{-3x}$$

$$\begin{aligned}\frac{dy}{dx} &= uv' + vu'\\ &= -3x^3e^{-3x} + 3x^2e^{-3x}\\ &= 3x^2e^{-3x}(1-x)\end{aligned}$$

(d) (3 marks)

$$\begin{aligned}y &= \log_e\left(\frac{2x+1}{3x-7}\right)\\ &= \log_e(2x+1) - \log_e(3x-7)\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{2x+1} - \frac{3}{3x-7}\\ &= \frac{-17}{(2x+1)(3x-7)}\end{aligned}$$

**2** (3 marks)

$$y = x\sin x$$
$$u = x \quad v = \sin x$$
$$u' = 1 \quad v' = \cos x$$

$$y' = x\cos x + \sin x\Big|_{x=\frac{\pi}{2}}$$
$$= \cancel{\tfrac{\pi}{2}\cos\left(\tfrac{\pi}{2}\right)} + \sin\left(\tfrac{\pi}{2}\right) = 1$$
$$\therefore m_{\perp} = -1$$

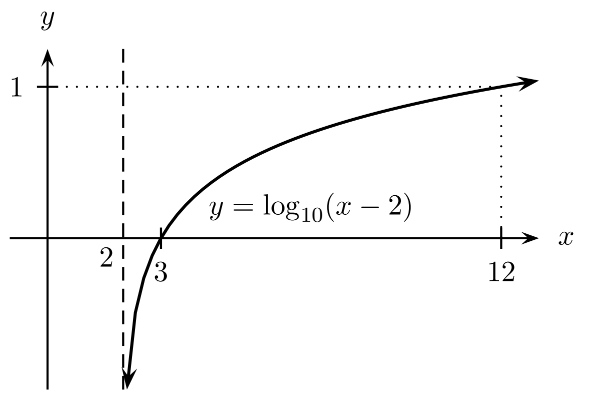
$$y = x\sin x\Big|_{x=\frac{\pi}{2}}$$
$$= \tfrac{\pi}{2}\sin\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{2}$$

At $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, $m_{\perp} = -1$. Using the point-gradient formula,

$$\frac{y - \frac{\pi}{2}}{x - \frac{\pi}{2}} = -1$$
$$y - \tfrac{\pi}{2} = \tfrac{\pi}{2} - x$$
$$+\tfrac{\pi}{2} \qquad +\tfrac{\pi}{2}$$
$$y = \pi - x$$

## Part C

**1** (3 marks)




$$D = \{x : x \in \mathbb{R}, x > 2\} \quad R = \{y : y \in \mathbb{R}\}$$

**2** (a) (2 marks)

$$\int \frac{2x}{x^2+1}\,dx = \log_e\left(x^2+1\right) + C$$

(b) (2 marks)

$$\int \frac{e^{2x}}{e^{2x}+4}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{e^{2x}+4}\,dx$$
$$= \frac{1}{2}\log_e\left(e^{2x}+4\right) + C$$

(c) (2 marks)

$$\int \frac{2}{x} + 5e^x\,dx = 2\log_e(x) + 5e^x + C$$

**3** (a) (3 marks)

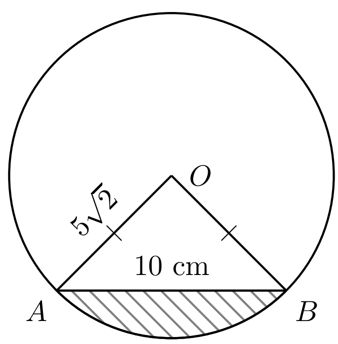
- 1 hr $\to 2\pi$
- 1 min $\to \frac{\pi}{30}$
- 35 min $\to 35 \times \frac{\pi}{30} = \frac{7\pi}{6}$

$$\ell = r\theta = 1.75 \times \frac{7\pi}{6} = 6.4 \text{ m}$$

(b) (2 marks)
Given $\frac{\pi}{30} \to 1$ min, then $\frac{\pi}{15} \to 2$ min.

## Part D

**1** (3 marks)

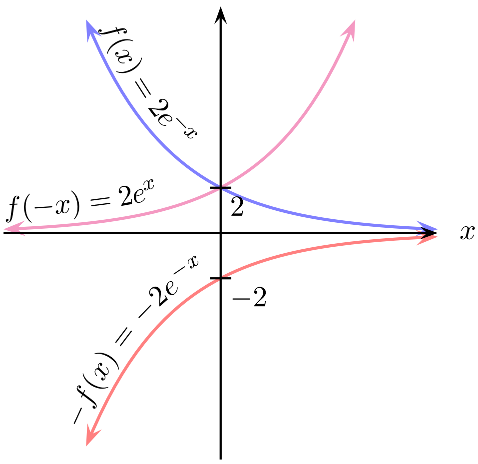



$$\cos\angle AOB = \frac{\left(5\sqrt{2}\right)^2 + \left(5\sqrt{2}\right)^2 - 10^2}{2 \times \left(5\sqrt{2}\right)\left(5\sqrt{2}\right)^2}$$
$$= \frac{50 + 50 - 100}{2 \times \left(5\sqrt{2}\right)\left(5\sqrt{2}\right)^2} = 0$$
$$\therefore \angle AOB = \frac{\pi}{2}$$

$$A = \frac{1}{2}r^2\left(\theta - \sin\theta\right)$$
$$= \frac{1}{2}\left(5\sqrt{2}\right)^2\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right)$$
$$= \frac{1}{2} \times 50\left(\frac{\pi}{2} - 1\right) = 25\left(\frac{\pi}{2} - 1\right) \text{ cm}^2$$

**2** (a) (2 marks) See next diagram – [1] for each correct curve.

(b) (1 mark)




**3** (a) (2 marks)

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} \cos 2x\,dx &= \frac{1}{2}\sin 2x\Big|_{\frac{\pi}{2}}^{\pi}\\ &= \frac{1}{2}\left(\cancel{\sin\left(2\cdot\frac{\pi}{2}\right)} - \cancel{\sin(2\cdot\pi)}\right)\\ &= 0\end{aligned}$$

(b) (2 marks)

$$\begin{aligned}&\int_0^{\frac{\pi}{4}} \frac{1}{2}x - \sin 2x\,dx\\ &= \frac{1}{4}x^2 + \frac{1}{2}\cos 2x\Big|_0^{\frac{\pi}{4}}\\ &= \frac{1}{4}\left(\left(\frac{\pi}{4}\right)^2 - \cancel{0^2}\right)\\ &\quad + \frac{1}{2}\left(\cancel{\cos\left(2\cdot\frac{\pi}{4}\right)} - \cancel{\cos 0}^{\,1}\right)\\ &= \frac{\pi^2}{64} - \frac{1}{2}\end{aligned}$$

## Part E

**1** (a) (2 marks)

$$\begin{aligned}y &= x\ln x - x\\ u &= x \quad v = \ln x\\ u' &= 1 \quad v' = \tfrac{1}{x}\\ \frac{dy}{dx} &= \cancel{x\cdot\tfrac{1}{x}} + \ln x - \cancel{1}\\ &= \ln x\end{aligned}$$

(b) (2 marks)

$$\begin{aligned}\int \ln\left(x^2\right)\,dx &= 2\int \ln x\,dx\\ &= 2\left(x\ln x - x\right) + C\\ &= 2x\ln x - 2x + C\end{aligned}$$

**2** (a) (4 marks)

$$y = \frac{e^x}{x^2+1}$$

$$\begin{aligned}u &= e^x \quad v = x^2+1\\ u' &= e^x \quad v' = 2x\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{vu' - uv'}{v^2}\\ &= \frac{e^x(x^2+1) - 2xe^x}{\left(x^2+1\right)^2}\\ &= \frac{e^x\left(x^2-2x+1\right)}{\left(x^2+1\right)^2}\\ &= \frac{e^x\left(x-1\right)^2}{\left(x^2+1\right)^2}\end{aligned}$$

| $x$ | $\frac{1}{2}$ | | $1$ | | $\frac{3}{2}$ |
|---|---|---|---|---|---|
| $\frac{dy}{dx}$ | | + | 0 | + | |
| $y$ | $\frac{4e^{1/2}}{5}$ | ↗ | $\frac{e}{2}$ | ↗ | $\frac{2e^{3/2}}{5}$ |

- Horizontal point of inflexion at $\left(1, \frac{e}{2}\right)$.

(b) (3 marks)

$$y = \frac{e^x}{x^2+1}$$

Since $e^x > 0$ and $x^2 + 1 > 0\ \forall x \in \mathbb{R}$, then

$$R = \{y : y \in \mathbb{R}, y > 0\}$$