THE SCOTS COLLEGE

THURSDAY, 5 JUNE 2008

ASSESSMENT 3

YEAR 12
MATHEMATICS

GENERAL INSTRUCTIONS

- Working time – 45 minutes.
- Attempt Questions 1 to 4.
- Start a new page for each Question.
- Board approved calculators may be used.
- All necessary working should be shown for every Question.

QUESTION	OUTCOME	MARK AVAILABLE	TOTAL	MARK OBTAINED	TOTAL
4d	H1	4	4		
3	H4	5	5		
1 2a 4a, c	H5	8 3 2, 3	16		
2b 4b	H9	3 2	5		
			30		

QUESTION 1 **[8 MARKS]**

The displacement of a particle is given by the equation $x = t^3 - 4t^2 - 3t$ where x is in metres and t is in seconds.

(a) Find the initial velocity. **[2]**

(b) Find when the particle is at rest. **[2]**

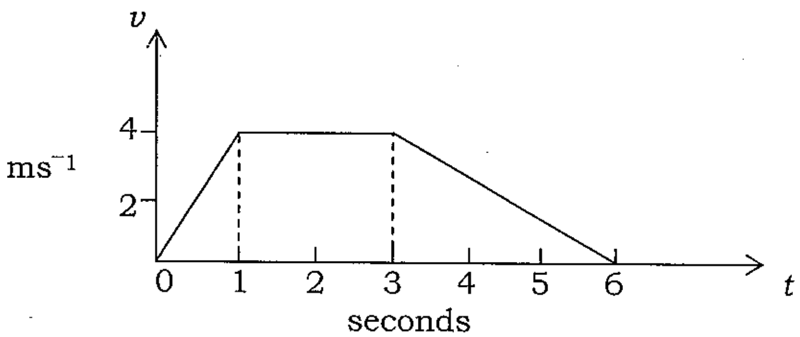
(c) Find the acceleration after 4 seconds. **[2]**

(d) Show that the particle is at the origin when $t = 0$ and $t = 2 + \sqrt{7}$. **[2]**

QUESTION 2 **[6 MARKS]**

(a) The acceleration of a particle is given by $a = -4\sin 2t$. Initially the particle is 4m to the left of the origin and the velocity is 2cms^{-1}. Find the displacement after $\frac{\pi}{4}$ seconds. **[3]**

(b) The velocity – time graph of a moving object is shown below.



(i) Find when the object is not subject to acceleration. **[1]**

(ii) Find the rate of deceleration when the object is slowing. **[1]**

(iii) Find the distance the object travels in the first three seconds. **[1]**

QUESTION 3 **[5 MARKS]**

A metal ball is cooling down according to the formula $T = T_0 e^{-kt}$ where T is the temperature (in degrees Celsius) and t is the time in minutes. The initial temperature of the ball is 50°C and it cools to 43°C after 15 minutes.

(a) Show that $k = 0.01$. **[2]**

(b) Find the temperature after 1 hour. **[1]**

(c) How long it takes to reach a room temperature of 21°C **[2]**

QUESTION 4 **[11 MARKS]**

(a) An arithmetic series has its sixth term equal to 18 and its eleventh term equal to 43. Find the series and write down the first three terms. **[2]**

(b) Evaluate $\sum_{n=1}^{n=8} 3^n$ **[2]**

(c) An employee earns $48,000 in their first year, with the wage increasing by 5% of the previous year's wage. Find:

(i) The employee's annual wage at the start of the sixth year. **[1]**

(ii) The total earnings (before tax) in the first five years. **[2]**

(d) A sum of $2,000 is invested at the start of every six months in a superannuation fund and accumulates every six months at an annual interest rate of 10%p.a.

(i) Find the value of the fund at the end of the first year. **[1]**

(ii) Show that at the end of n years, the fund has grown to $\$40{,}000(1.05)\left(1.05^{2n}-1\right)$. **[3]**

Standard integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

SOLUTIONS

$x = t^3 - 4t^2 - 3t$

$v = \frac{dx}{dt} = 3t^2 - 8t - 3$

at $t = 0$, $v = -3$ m/s

(b) The particle is at rest when $v = 0$

$\therefore 3t^2 - 8t - 3 = 0$

$(3t+1)(t-3) = 0$

$\therefore t = 3, -\frac{1}{3}$

ie $t = 3$ since $t \geqslant 0$

(c) $a = \frac{dv}{dt} = 6t - 8$

at $t = 4$, $a = 6(4) - 8 = 16$ m/s²

(d) $x = t^3 - 4t^2 - 3t$

$= t(t^2 - 4t - 3)$

when $x = 0$, $t = 0$ and $t^2 - 4t - 3 = 0$.

Solving $t^2 - 4t - 3 = 0$, $t = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$

$= \frac{4 \pm \sqrt{16+12}}{2}$

$= \frac{4 \pm \sqrt{28}}{2}$

$= \frac{4 \pm 2\sqrt{7}}{2}$

$= 2 \pm \sqrt{7}$

Since $t \geqslant 0$, $t = 2 + \sqrt{7}$.

ie. at $x = 0$, $t = 0, 2+\sqrt{7}$ seconds.

$a = -4\sin 2t$

$v = \int(-4\sin 2t)\,dt$

$v = 2\cos 2t + c_1$ $\quad \frac{1}{2}$

at $t=0$, $v=2$.

$\therefore 2 = 2\cos 0 + c_1$ $\quad$ 1

$2 = 2 + c_1 \quad \therefore c_1 = 0$

$v = 2\cos 2t$ $\quad \frac{1}{2}$

$x = \int 2\cos 2t\,dt$

$x = \sin 2t + c_2$ $\quad \frac{1}{2}$

at $t=0$, $x=-4$.

$-4 = \sin 0 + c_2 \quad \therefore c_2 = -4$ $\quad \frac{1}{2}$ $\quad$ 1

$x = \sin 2t - 4$

at $t = \frac{\pi}{4}$, $x = \sin 2\left(\frac{\pi}{4}\right) - 4$

$= \sin \frac{\pi}{2} - 4$

$= 1 - 4$ $\quad$ 1

$= -3$ m.

(b) (i) The object is not accelerating when the velocity is constant. This occurs between 1 and 3 seconds.

(ii) The object decelerates between 3 s and 6 s. from 4 ms^{-1} to 0 ms^{-1} $\quad$ 1

$\therefore$ rate deceleration $= \frac{\text{speed}}{\text{time}} = \frac{4-0}{6-3}$

$= \frac{4}{3}\ ms^{-2}$ $\quad$ m/s^2 $\quad$ 1

(iii) $x = \int v\,dt$

$\equiv$ area from 0 s to 3 s.

$= \frac{1}{2}4(2+3)$ $\quad$ (area of trapezium) $\quad$ 1

$= 10$ m

$T = T_0 e^{-kt}$

at $t = 0$, $T = 50$.

$\therefore 50 = T_0 e^{-k(0)}$

$50 = T_0 (1)$

$T_0 = 50$

(ii) $T = 50 e^{-kt}$

$43 = 50 e^{-15k}$

$e^{-15k} = \frac{43}{50} = 0.86$

$-15k = \ln 0.86$

$k = \frac{-\ln 0.86}{15} = 0.0100$

For $t = 60$,

$T = 50 e^{-(0.01)60}$

$= 27.44°C$

(iii) $T = T_0 e^{-kt}$

$21 = 50 e^{-(0.01)t}$

$e^{-0.01t} = \frac{21}{50}$

$\therefore -0.01t = \ln \frac{21}{50}$

$t = \frac{\ln 0.42}{-0.01}$ mins

$= 86.75$ mins

Q 4.

(a) $T_6 = 18 = a + 5d$

$T_{11} = 43 = a + 10d$

$\therefore 5d = 25$

$d = 5$.

$18 = a + 5(5)$

$\therefore a = -7$.

Series is $-7, -2, 3$

(b) $\sum_{n=1}^{n=8} 3^n$

$= \underbrace{3^1 + 3^2 + \ldots 3^8}_{S_n = \frac{a(r^n - 1)}{r-1}}$

$= \frac{3(3^8 - 1)}{3 - 1}$

$= \frac{3}{2}(3^8 - 1)$

$= \frac{3}{2}(6561 - 1)$

$= 9840$

(i) $A = P\left(1+\frac{r}{100}\right)^n$ $\quad P = 48,000$, $r = 5\%$ pa, $n = 5$ yrs.

$= 48,000\left(1+\frac{5}{100}\right)^5$

$= 48,000(1.05)^5$

$= \$$

(ii) Total earnings $: 48,000 + 48000(1.05) + 48000(1.05)^2 + \cdot \quad 48000(1.05)$

$= 48000\left(1+1.05+1.05^2+\ldots 1.05^4\right)$

$= 48000\left[\frac{1(1.05^5-1)}{1.05-1}\right]$ $\quad S_n = a\left(\frac{r^n-1}{r-1}\right)$

$= \frac{48,000(1.05^5-1)}{0.05}$

$= \$265,230$

~~(c) $\therefore A_n = P\left(1+\frac{r}{100}\right)^n$~~

~~Let A_n be the amount accumulating after n years.~~

~~$A_{40} = 2000(1.12)^{40}$~~

~~$A_{39} = 2000(1.12)^{39}$~~

~~$A_2 = 2000(1.12)^2$~~

~~$A_1 = 2000(1.12)^1$~~

~~$\therefore$ Total amount $= 2000\left[1.12+1.12^2+ \ldots 1.12^{40}\right]$~~

~~$= 2000\left[\frac{1.12(1.12^{40}-1)}{1.12-1}\right]$~~

~~$= \frac{2000(1.12)(1.12^{40}-1)}{0.12}$~~

~~$= \$1,718,284.78$~~

Let A_n be amount of n ha... ye...

(i) $A_1 = 2000(1.05)$

$A_2 = 2000(1.05) + 2000(1.0\ldots$

$= 2000(1.05+1.05^2)$

$=$

(ii) $A_3 = 2000(1.05+1.05^2+1.05\ldots$

Continuing this pattern

$A'_{2n} = 2000(1.05+\ldots 1.0\ldots$

$= \frac{2000(1.05)(1.05^{2n}-1)}{.05}$

$= 40,000(1.05)(1.05^{2n}-1)$