SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

JUNE 2005

TASK #3

YEAR 12

Mathematics

General Instructions

- Reading time – 5 minutes.
- Working time – 2 Hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each **NEW** section in a separate answer booklet.

Total Marks - 100 Marks

- Attempt Questions 1 - 6
- All questions are NOT of equal value.

Examiner: *P. Bigelow*

Total marks – 100
Attempt Questions 1 - 6
All questions are NOT of equal value

Answer each SECTION in a SEPARATE writing booklet.

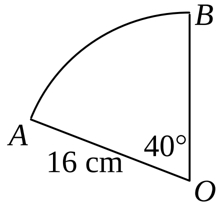
Section A

Marks

Question 1 (15 marks)

(a) Convert $150°$ to radians **1**

(b) **2**



OAB is a sector of a circle, with radius 16 cm.

Write down the area of the sector *OAB*, correct to the nearest square centimetre.

(c) Write down the value of the following, correct to 2 decimal places.

(i) $\frac{1}{\sqrt{e}}$ **1**

(ii) $\cos 3$ **1**

(d) Write down the derivatives of the following

(i) e^{-4x} **1**

(ii) $\tan\left(\frac{x}{2}\right)$ **1**

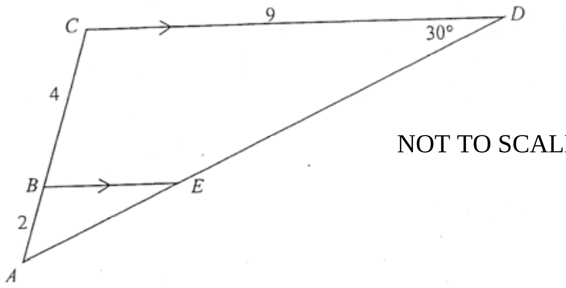
(e) Thirty cards are numbered from 1 to 30.

(i) If one card is selected at random what is the probability that the card has a **4** on it? **1**

(ii) If two cards are chosen at random, what is the probability that at least one of the cards has a **4** on it? **2**

(f) Find $f(2)$ if $f'(x) = 2x + 3$ and $f(1) = 6$ **2**

(g)



In the diagram, ACD is a triangle where AB = 2 cm, BC = 4 cm, CD = 9 cm and $\angle CDE = 30°$.
Also BE is parallel to CD.

(i) Find the size of $\angle BED$. **1**

(ii) Find the length of BE. Give reasons for your answer. **2**

Question 2 (17 marks)

(a) Differentiate the following

(i) $2x\ln x$ **2**

(ii) $\dfrac{\cos x}{x}$ **2**

(b) Find

(i) $\displaystyle\int_1^4 \frac{dx}{\sqrt{x^3}}$ **2**

(ii) $\displaystyle\int \frac{x^3+x}{x^2}\,dx$ **2**

(iii) $\displaystyle\int \left(e^x+1\right)^2 dx$ **2**

Question 2 continues over the page

Question 2 continued

(c) A farmer made the following statement:
"*In* 1998 *the price of wheat started to fall and it reached its lowest levels in mid* 2002*. Then from mid* 2002*, the price began a continual rise and is now, in* 2005*, at an all time high.*"

If the curve representing the price of wheat, P, is differentiable, what does the above statement imply about:

(i) $\dfrac{dP}{dt}$ between 1998 and 2005 **2**

(ii) $\dfrac{d^2P}{dt^2}$ at mid 2002? **1**

(d) (i) Sketch $y = 3\sin 2x$ for $0 \le x \le 2\pi$ **2**

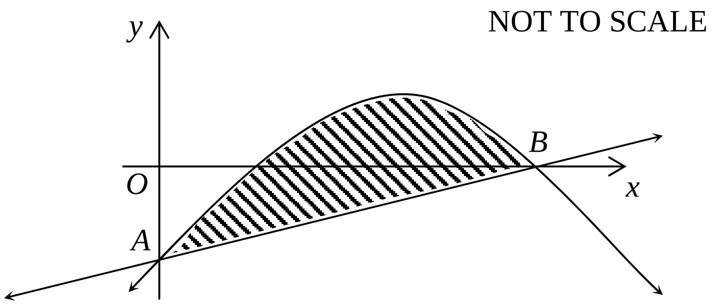
(ii) Find $\displaystyle\int_0^{\frac{\pi}{4}} 3\sin 2x \, dx$ **2**

End of Section A

Section B (Use a SEPARATE writing booklet)

Question 3 (17 marks) **Marks**

(a) The graphs of $y = x - 4$ and $y = -x^2 + 5x - 4$ intersect at the points *A* and *B*, as shown in the diagram below.



(i) Find the coordinates of *A* and *B* **2**

(ii) Find the area of the shaded part. **2**

(b) Find the equation of the tangent to $y = \ln x$ at the point where $y = \ln x$ crosses the *x* axis. **2**

(c) Consider the curve given by $y = x^3 - 6x^2 + 9x + 6$

(i) Find the coordinates of the stationary points and determine their nature. **3**

(ii) Find the coordinates of any points of inflexion. **1**

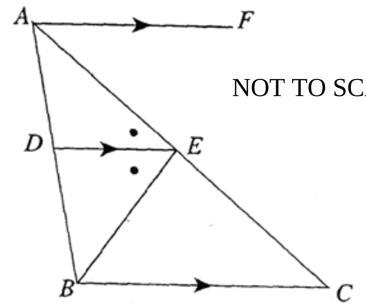
(iii) Sketch the curve for $-1 \le x \le 4$. **2**

(iv) Indicate on your curve, with the letter *S*, where the curve has the greatest rate of increase. **1**

Question 3 is continued over the page

Question 3 continued

(d)



The diagram shows ΔABC with *D* lying on *AB* and *E* lying on *AC*.
The lines *AF*, *DE* and *BC* are parallel and $\angle AED = \angle BED$.

(i) Show that ΔBEC is isosceles. **2**

(ii) State the reason why $AD : DB = AE : EC$. **1**

(iii) Show that $AD : DB = AE : EB$ **1**

Question 4 (16 marks)

(a) The part of the curve $y = \dfrac{1}{\sqrt{2x+1}}$ between $x = 0$ and $x = 1$ is rotated about the *x* axis. **3**

Find the volume of the solid obtained

(b) Five marbles are numbered 1, 2, 3, 4 and 5 and placed in bag. Two marbles are taken out in succession, the first marble not being replaced before the second is withdrawn.

(i) Find the probability that

(α) the 4 is selected. **1**

(β) the 3 is NOT selected. **1**

(γ) the 1 is the second marble chosen. **1**

(ii) If the first marble is replaced before the second is selected, find the probability that at least one 2 is drawn. **2**

Question 4 continued

(c) (i) Differentiate $y = \ln(\cos x)$ **2**

(ii) Hence find $\int \tan x \, dx$ **1**

(d) Find $\int_0^{\frac{\pi}{6}} \sec^2 2x \, dx$ **2**

(e) For $a > 0$, it is given that **3**

$$\int_1^4 \frac{4x}{4+x^2} \, dx = \ln a$$

By evaluating the integral, write down the value of a.

End of Section B

Section C (Use a SEPARATE writing booklet)

Question 5 (16 marks) **Marks**

(a) Find the second derivative of $x\sin x$. **2**

(b) Find $\int_0^1 \frac{e^x}{1+e^x}\,dx$ **2**

(c) A continuous curve $y = f(x)$ has the following properties over the interval $a \le x \le b$:

$$f(x) > 0,\ f'(x) > 0,\ f''(x) > 0$$

(i) Sketch a curve satisfying these properties. **2**

(ii) State the least value of $f(x)$ over this interval. **1**

(d) At every point on a curve $\frac{d^2y}{dx^2} = 2x$. **4**

The point $(3,6)$ lies on the curve and its tangent at this point is inclined at $45°$ to the positive direction of the x axis.

Find the equation of the curve

(d) (i) For the function $f(x) = e^{x^2}$, copy and complete the following table, using 4 significant figures where necessary. **2**

x	-1	$-0\cdot5$	0	$0\cdot5$	1
$f(x)$					

(ii) Hence, use Simpsons rule with 5 function values to approximate **3**

$$\int_{-1}^{1} e^{x^2}\,dx\ .$$

Leave your answer correct to 3 significant figures.

Question 6 (19 marks) **Marks**

(a) (i) Sketch $y = \log_e x$ and $y = x$ on the same diagram. **2**

(ii) Use your diagram in (i) to indicate the number of solutions to the equation $\log_e x = x$ **1**

(b) Given the curve $y = 2xe^{\frac{x}{2}}$

(i) Show that $\frac{dy}{dx} = (x+2)e^{\frac{x}{2}}$ **2**

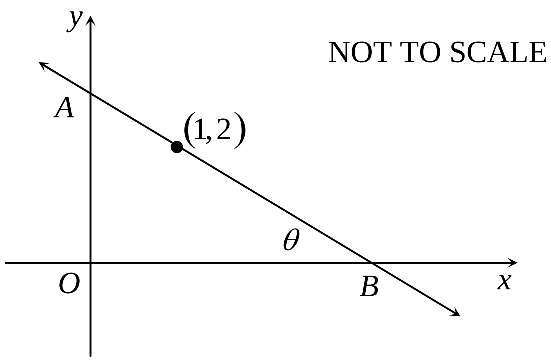
(ii) Find $\frac{d^2y}{dx^2}$ **2**

(iii) Find the minimum value of $2xe^{\frac{x}{2}}$ **2**

(iv) For what values of x is the curve concave down? **1**

(v) For what values of c does the equation $2xe^{\frac{x}{2}} = c$ have two unequal real roots? **1**

(c) In the diagram AB meets the x and y axes at B and A respectively and it passes through the point $(1,2)$.
The angle OBA is θ in radians.



(i) Show that the equation of AB is given by **2**

$$y = -(\tan\theta)x + 2 + \tan\theta$$

(ii) Find the area of ΔOAB in terms of $\tan\theta$. **2**

(iii) Find the value of θ, correct to 2 decimal places, for which the area is a minimum. **4**

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a} \sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a} \cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

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SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2005
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #3

Mathematics

Sample Solutions

Section	Marker
A	AF
B	RB
C	RD

Question 1

(a) $150 \times \frac{\pi}{180} = \frac{5\pi}{6}^c$

(b) $\theta = 40 \times \frac{\pi}{180} = \frac{2\pi}{9}^c$

$\text{Area} = \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times (16)^2 \times \frac{2\pi}{9}$

$= \frac{256\pi}{9} \text{ cm}^2$

$\approx 89 \text{ cm}^2$

(c)(i) $\frac{1}{\sqrt{e}} \approx 0.61$

(ii) $\cos 3 = -0.99$

(d)(i) $-4e^{-4x}$

(ii) $\frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

(e)(i) $P(\text{no 4's}) = \frac{27}{30} \times \frac{26}{29}$

$= \frac{117}{145}$

$P(\text{at least one 4}) = 1 - \frac{117}{145}$

$= \frac{28}{145}$

(f) $f'(x) = 2x + 3$

$f(x) = x^2 + 3x + c$

since $f(1) = 6$

$6 = (1)^2 + 3(1) + c$

$c = 2$

$f(x) = x^2 + 3x + 2$

$f(2) = (2)^2 + 3(2) + 2$

$= 12$

(g)(i) $\angle BED = 150°$

(co-interior angles are supplementary, $BE \parallel CD$)

(ii) In $\triangle$'s BEA & CDA

$\angle CAE$ is common

$\angle BEA = \angle CDA$ (corresponding angles, $BE \parallel CD$)

$\therefore \triangle BEA \parallel\!\!| \triangle CDA$

$\therefore$ corresponding sides are in the same ratio

$\frac{BE}{CD} = \frac{AB}{AC}$

$\frac{BE}{9} = \frac{2}{6}$

$BE = 3 \text{ cm}$

Question 2

(a)(i) $y' = uv' + vu'$ $\quad u = 2x,\ v = \ln x,\ u' = 2,\ v' = \frac{1}{x}$

$y' = 2x \cdot \frac{1}{x} + 2 \cdot \ln x$

$= 2 + 2\ln x$

(ii) $y' = \frac{vu' - uv'}{v^2}$ $\quad u = \cos x,\ v = x,\ u' = -\sin x,\ v' = 1$

$y' = \frac{x(-\sin x) - 1 \cdot \cos x}{x^2}$

$= \frac{-x\sin x - \cos x}{x^2}$

(b)(i) $\int_1^4 x^{-3/2}\,dx$

$= \left[-2x^{-\frac{1}{2}}\right]_1^4$

$= -2(4)^{-\frac{1}{2}} - \left(-2(1)^{-\frac{1}{2}}\right)$

$= -1 + 2$

$= 1$

(ii) $\int \frac{x^3 + x}{x^2}\,dx$

$= \int \left(x + \frac{1}{x}\right) dx$

$= \frac{x^2}{2} + \ln x + C$

(iii) $\int (e^{2x} + 2e^x + 1)\,dx$

$= \frac{1}{2}e^{2x} + 2e^x + x + C$

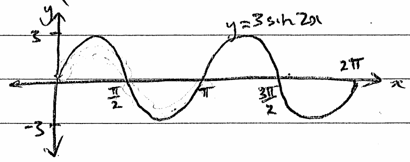
(c)(i) $\frac{dp}{dt} < 0$ from 1998 to mid 2002

$\frac{dp}{dt} = 0$ in mid 2002

$\frac{dp}{dt} > 0$ from mid 2002 to now

(ii) $\frac{d^2p}{dt^2} > 0$

(d)(i) amplitude = 3 $\quad$ period $= \frac{2\pi}{2} = \pi$



(ii) $\int_0^{\frac{\pi}{4}} 3\sin 2x\,dx$

$= \left[-\frac{3}{2}\cos 2x\right]_0^{\frac{\pi}{4}}$

$= -\frac{3}{2}\cos 2\left(\frac{\pi}{4}\right) - \left(-\frac{3}{2}\cos 0\right)$

$= ~~-\frac{3}{2}\cos\frac{\pi}{2}~~ + \frac{3}{2}\cos 0$

$= \frac{3}{2}$

3) (a)(i) $y = x - 4$

$y = -x^2 + 5x - 4$

let $x - 4 = -x^2 + 5x - 4$

$x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0 \quad y = -4 \quad A(0, -4)$

$x = 4, \quad y = 0 \quad B(4, 0)$

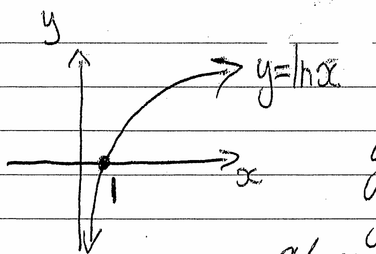
(ii) $A = \int_0^4 \left[(-x^2 + 5x - 4) - (x - 4)\right] dx$

$= \int_0^4 (-x^2 + 4x)\, dx$

$= -\frac{x^3}{3} + 2x^2 \Big]_0^4 = -\frac{4^3}{3} + 2 \times 4^2 - 0$

$= -\frac{64}{3} + 32 = 10\frac{2}{3}\ u^2$ //

(b)



Crosses x axis at (1, 0).

$y = \ln x$

$y' = \frac{1}{x}$

at $x = 1$, $m = \frac{1}{1} = 1$

using $(y - y_1) = m(x - x_1)$

$y - 0 = 1(x - 1)$

$y = x - 1$ // OR $x - y - 1 = 0$

(c) $y = x^3 - 6x^2 + 9x + 6$

$y' = 3x^2 - 12x + 9$

$y'' = 6x - 12$

(i) Stat pts exist when $y' = 0$ $3x^2 - 12x + 9 = 0$

$3(x^2 - 4x + 3) = 0$

$x^2 - 4x + 3 = 0$

$(x - 3)(x - 1) = 0$

$x = 3$ $y = 27 - 54 + 27 + 6 = 6$ $(3, 6)$

$x = 1$ $y = 1 - 6 + 9 + 6 = 10$ $(1, 10)$

At $(3, 6)$ $y'' = 6 \times 3 - 12 > 0$ min stat pt.

At $(1, 10)$ $y'' = 6 \times 1 - 12 < 0$ max stat pt.

(ii) inflexions occur when $y'' = 0$ and there is a sign change.

$6x - 12 = 0$

$6(x - 2) = 0$

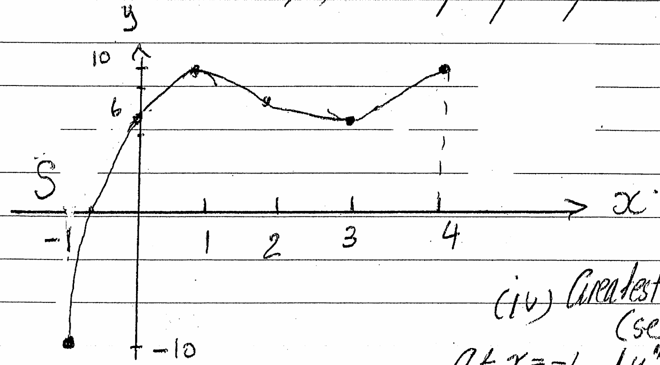
$x = 2$, $y = 8 - 6 \times 4 + 18 + 6 = 8$ $(2, 8)$.

At $(2, 8)$ $x = 2 - \varepsilon$ $y'' < 0$ } sign change

$x = 2 + \varepsilon$ $y'' > 0$ }

$(2, 8)$ is a pt of inflexion

(iii)



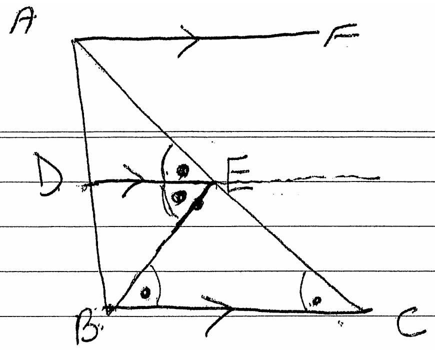
At $x = -1$, $y = -1 - 6 - 9 + 6$
$= -10$

At $x = 4$ $y = 64 - 96 + 36 + 6$
$= 10$

(iv) Greatest rate of increase is (see diagram)

At $x = -1$ $|y'|$ is greatest.

3 (d)



(i) $E\hat{B}C = B\hat{E}D$ alternate angles $DE \parallel BC$
$B\hat{C}E = A\hat{E}D$ corresponding angles $DE \parallel BC$
$\therefore$ $\triangle BEC$ is isosceles, $E\hat{B}C = B\hat{C}E$.

(ii) Why does $\frac{AD}{DB} = \frac{AE}{EC}$ ratio of intercepts result, as $AF \parallel DE \parallel BC$ and AC is the transversal.

(iii) Since $BE = EC$ from (i)
$\frac{AD}{DB} = \frac{AE}{EC}$ from (ii)

So $\frac{AD}{DB} = \frac{AE}{EB}$

4 (a) $V = \pi \int_0^1 \left(\frac{1}{\sqrt{2x+1}}\right)^2 dx$

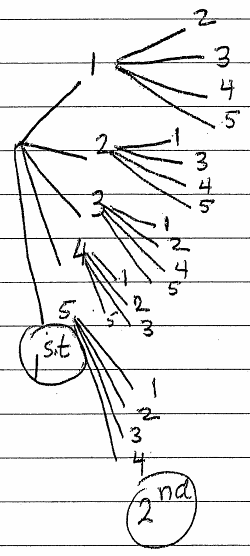
$= \pi \int_0^1 \frac{1}{2x+1} \, dx$

$= \frac{\pi}{2} \ln(2x+1) \Big]_0^1$

$= \frac{\pi}{2} \left[\ln(3) - \ln(1)\right]$

$= \frac{\pi}{2} \ln 3 \; u^3$ or $\doteqdot 1{\cdot}73 \, u^3$

(b)



(i) (α) $1 - \frac{12}{20} = \frac{4}{10} = \frac{2}{5}$

(β) $\frac{12}{20} = \frac{3}{5}$

(γ) $\frac{4}{20} = \frac{1}{5}$

(ii) P(at least one 2) = one 2 on either marble or both 2

$\frac{9}{25}$

(c) (i) $y = \ln(\cos x)$

$y' = \frac{1}{\cos x} \times -\sin x = \frac{-\sin x}{\cos x} = -\tan x$

(ii) $-\int -\tan x \, dx = -\ln(\cos x) + C$

(d)

$\frac{1}{2}\int_0^{\frac{\pi}{6}} 2\sec^2 2x \, dx = \frac{1}{2}\tan 2x \Big]_0^{\frac{\pi}{6}}$

$= \frac{1}{2}\left(\tan\frac{\pi}{3} - \tan 0\right)$

$= \frac{1}{2}(\sqrt{3} - 0)$

$= \frac{\sqrt{3}}{2}$ or $\doteqdot 0.866$

(e) $\int_1^4 \frac{4x}{4+x^2} \, dx$

$\frac{4}{2}\int_1^4 \frac{2x}{4+x^2} \, dx$

$= 2\ln(4+x^2)\Big]_1^4$

$= 2\ln 20 - 2\ln 5$

$\Rightarrow 2[\ln 20 - \ln 5]$

$= 2\ln\left(\frac{20}{5}\right)$

$= 2\ln 4$

$= \ln 4^2 = \ln 16$

So $a = 16$ //

Q5(a) $y = x \sin x$

$y' = \sin x \cdot 1 + x \cdot \cos x$

$= \sin x + x \cos x$

$y'' = \cos x + (\cos x \cdot 1 + x \cdot -\sin x)$

$= 2\cos x - x \sin x$

(b) $\int_0^1 \frac{e^x}{1+e^x}\,dx$

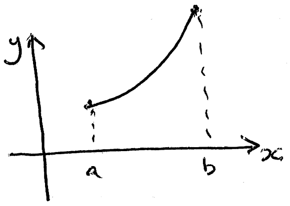
$= \left[\ln(1+e^x)\right]_0^1$

$= \ln(1+e) - \ln(1+1)$

$= \ln(1+e) - \ln 2$

$= \ln\left(\frac{1+e}{2}\right)$

c) (i)



(ii) $f(a)$

(d) $y'' = 2x$

$y' = x^2 + c$

when $x=3$: $1 = 9 + c$

$\therefore c = -8$

$\therefore y' = x^2 - 8$

$y = \frac{1}{3}x^3 - 8x + c_1$

When $x=3$: $6 = 9 - 24 + c_1$

$c_1 = 21$

$\therefore y = \frac{1}{3}x^3 - 8x + 21$

(d) $f(x) = e^{x^2}$

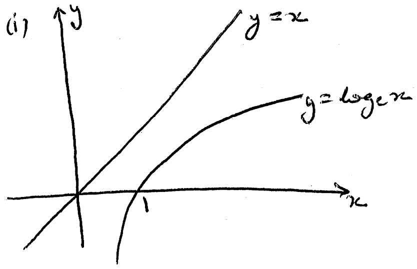
x	-1	-0.5	0	0.5	1
$f(x)$	2.718	1.284	1	1.284	2.718

(ii) $\int_{-1}^1 e^{x^2}\,dx \approx \frac{0.5}{3}(2.718 + 4\times 1.284 + 2\times 1 + 4\times 1.284 + 2.718)$

$= 2.9513333\ldots$

≈ 2.95

Q6 (a) (i)



(ii) 0

(b) (i) $y = 2x . e^{x/2}$

$y' = e^{x/2} . 2 + 2x . e^{x/2} . \frac{1}{2}$

$= e^{x/2} . 2 + x . e^{x/2}$

$= (x+2) e^{x/2}$

(ii) $y'' = e^{x/2} . 1 + (x+2) . e^{x/2} . \frac{1}{2}$

$= e^{x/2} + \frac{1}{2} x e^{x/2} + e^{x/2}$

$= e^{x/2} (\frac{1}{2} x + 2)$

(iii) For min value, consider $y' = 0$

$\therefore x = -2$

When $x = -2$ $y'' = e^{-1}(-1+2)$

$= e^{-1} > 0$

$\therefore$ Min value when $x = -2$

$\therefore$ Min value is $2 \times -2 \times e^{-1}$

$= -4(e^{-1})$ $(\approx -1.4715..)$

(iv) Concave down $\Rightarrow y'' < 0$

$\therefore \frac{1}{2} x + 2 < 0$

$\therefore \frac{1}{2} x < -2$

$\therefore x < -4$

(v) 2 unequal roots if $-4e^{-1} < c < 0$

(c) (i) $m = \tan(\pi - \theta)$

$= -\tan\theta$

$\therefore$ Eqn is $y - 2 = -\tan\theta (x - 1)$

$\therefore y - 2 = -\tan\theta\, x + \tan\theta$

$\therefore y = -\tan\theta\, x + 2 + \tan\theta$

(ii) If $x = 0$, $y = 2 + \tan\theta \Rightarrow A$ is $(0, 2+\tan\theta)$

If $y = 0$, $\tan\theta\, x = 2 + \tan\theta$

$x = \frac{2+\tan\theta}{\tan\theta} \Rightarrow B$ is $(\frac{2+\tan\theta}{\tan\theta}, 0)$

$\therefore$ Area of $\Delta AOB = \frac{1}{2} \times \frac{2+\tan\theta}{\tan\theta} (2 + \tan\theta)$

$= \frac{(2+\tan\theta)^2}{2\tan\theta}$

(iii) $\frac{dA}{d\theta} = \frac{(2\tan\theta . 2(2+\tan\theta)\sec^2\theta - (2+\tan\theta)^2 . 2\sec^2\theta)}{4\tan^2\theta}$

$= \frac{2\sec^2\theta (2+\tan\theta)(2\tan\theta - (2+\tan\theta))}{4\tan^2\theta}$

$= \frac{2\sec^2\theta (2+\tan\theta)(\tan\theta - 2)}{4\tan^2\theta}$

$= \frac{(2+\tan\theta)(\tan\theta - 2)}{2\sin^2\theta}$

Consider $\frac{dA}{d\theta} = 0$

$\therefore \tan\theta = 2$ or $\tan\theta = -2$

$\therefore \tan\theta = 2$ (as $0 < \theta < \frac{\pi}{2}$)

$\therefore \theta = 1.107148...$

θ	1	1.107	2
$\frac{dA}{d\theta}$	-ve	0	+ve
Curve	\	—	/

$\therefore$ Min value occurs when $\theta \approx 1.11$