SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

JUNE 2008
TASK #3
YEAR 12

Mathematics

General Instructions:

- Reading time—5 minutes.
- Working time—2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Start each NEW section in a separate answer booklet.
- Hand in your answer booklets in 3 sections:
  Section A(Questions 1 and 2),
  Section B(Questions 3 and 4),
  Section C(Questions 5 and 6),

Total marks—90 Marks

- Attempt questions 1–6.
- All questions are of equal value.

Examiner: Mr D. Hespe

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Section A

Marks

Question 1 (15 marks)

(a) Find primitives of the following differential functions:

(i) $f'(x) = 2x - 3$ **1**

(ii) $f'(x) = x^2 - 2x + 1$ **1**

(iii) $f'(x) = 0$ **1**

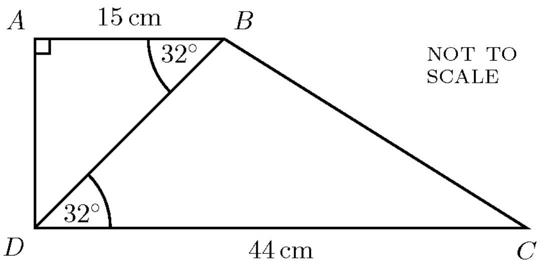
(iv) $f'(x) = (3x + 1)(x - 2)$ **2**

(b) $ABCD$ is a quadrilateral where $A = (-3, 0)$, $B = (0, 4)$, $C = (5, 4)$, and $D = (2, 0)$.

(i) Show that $ABCD$ is a rhombus. **3**

(ii) Prove that its diagonals are perpendicular bisectors of one another. **2**

(c)



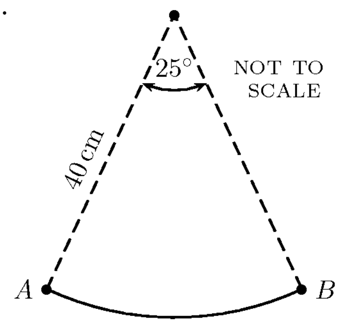
(i) Prove that $ABCD$ is a trapezium. **2**

(ii) Calculate the length of BC to a suitable degree of accuracy. **3**

Marks

Question 2 (15 marks)

(a) A pendulum 40 cm long swings through an angle of 25°. Find the length of the arc AB correct to the nearest millimetre. **2**



(b) Find the exact value of

(i) $\sin \frac{\pi}{4}$ **1**

(ii) $\cot \frac{4\pi}{3}$ **1**

(iii) any angles where $\cos\theta = \frac{\sqrt{3}}{2}, \quad -\pi \leqslant \theta \leqslant \pi$. **2**

(c) Find

(i) $\frac{d}{dx}\sin(3x+\pi)$ **1**

(ii) $\frac{d}{dx}\log_e(x^2 - x + 2)$ **1**

(iii) $\frac{d}{dx}\left(e^{2x} - e^{x} - e^{-x}\right)$ **1**

(iv) $\frac{d}{d\theta}(\ln\tan\theta)$ **2**

(d) Calculate, correct to 2 decimal places:

(i) $\sin 2$ **1**

(ii) $\ln 17$ **1**

(e) Find the second derivative of e^{x^2}. **2**

Section B

(Use a separate writing booklet.)

Marks

Question 3 (15 marks)

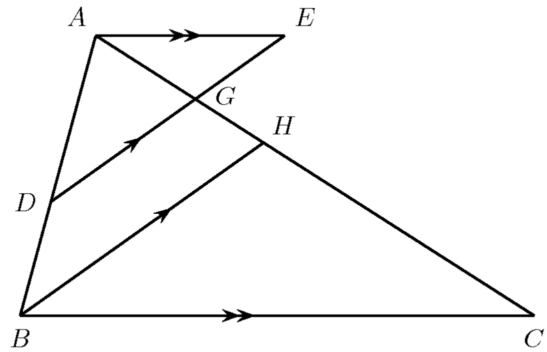
(a) Show that

(i) $\displaystyle\int_0^{\pi/4} \cos 2x \, dx = 0.5,$ 2

(ii) $\displaystyle\int_2^7 \left| 3 - \frac{x}{2} \right| dx = 4.25.$ 3

(b) Find the equations of the circles which pass through the points (3, 0) and (12, 0) and are tangent to the y-axis. 4

(c)



In the diagram (NOT TO SCALE) AE is parallel to BC and $AE = \frac{2}{7}BC$. The point D on AB is such that $BD = \frac{3}{7}BA$. The line DE meets AC at G and the line through B parallel to DE meets AC at H.

(i) Prove that $A\widehat{G}E = B\widehat{H}C$. 3

(ii) Prove that $\triangle AEG$ is similar to $\triangle CBH$. 3

Marks

Question 4 (15 marks)

(a) Find in radians all solutions to $\sin x = 0.8$ with $0 \leqslant x \leqslant 2\pi$. Give your answers to 4 decimal places. **2**

(b) Evaluate $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx$ by

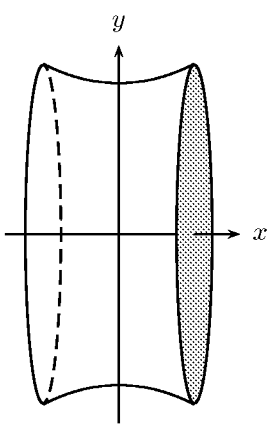
(i) direct integration, **2**

(ii) the trapezoidal rule using two strips, **2**

(iii) Simpson's rule using three function values. **2**

(iv) By means of a sketch or otherwise, explain why part (iii) gives a closer approximation to part (i) than part (ii). **2**

(c)



Find the exact volume of a metal component of a yacht's rigging system that is made in the shape of the solid formed when the curve $y = 2(e^{0.5x} + e^{-0.5x})$ between $x = -1$ and $x = 1$ is revolved around the x-axis. **5**

Section C

(Use a separate writing booklet.)

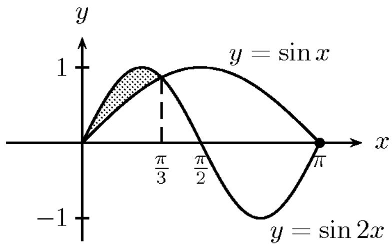
Marks

Question 5 (15 marks)

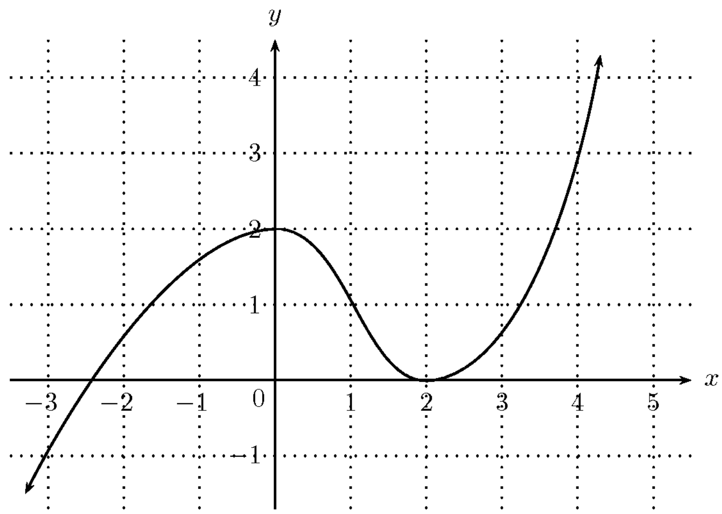
(a) (i) Sketch the region bounded by $y = 1$, $x = 8$, and the curve $y = \dfrac{1}{x}$. **2**

(ii) Determine the area of this region with the aid of an appropriate integral. **2**

(b) Calculate the area of the shaded region in the sketch below. **3**



(c) The curve shown is a differential function, $y = f'(x)$. Copy the curve onto your answer booklet and, on the same axes, sketch a possible $y = f(x)$. **4**



(d) Given that $10^x = e^{x \ln 10}$, find

(i) $\dfrac{d}{dx}\left(10^x\right)$, **2**

(ii) $\displaystyle\int 10^x \, dx$. **2**

Marks

Question 6 (15 marks)

(a) A hemispherical bowl of radius r cm is to be partly filled with water to a depth of h cm.

(i) What is the domain of h? 1

(ii) Draw a clear diagram with an appropriately labelled set of axes. 1

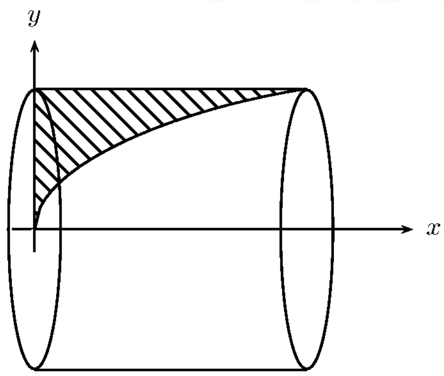
(iii) By using the method of volume of revolution, find the volume of water when the depth is h. 3

(b) (i) Differentiate $x \ln x$ and express in simplest form. 1

(ii) Hence find $\int \ln x \, dx$. 2

(c) The gradient of a curve at any point on it is $\dfrac{2}{2x+1}$ and the curve passes through the point $(1, \log_e 3)$. Find the equation of the curve. 3

(d) The area bounded by the curve $y = \sqrt{\sin x}$, the line $y = \dfrac{1}{\sqrt{2}}$, and the y-axis is rotated about the x-axis. Find the exact volume of this solid of revolution. 4



End of Paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

June 2008 2U Maths Assess Task 3.

Section A

1 (a) (i) $\int (2x-3)\,dx$

$= \frac{2x^2}{2} - 3x + C$

$= x^2 - 3x + C$ (1)

(ii) $\int (x^2-2x+1)\,dx$

$= \frac{x^3}{3} - \frac{2x^2}{2} + 1x + C$

$= \frac{x^3}{3} - x^2 + x + C$ (1)

(iii) $\int 0\,dx$

$= C$ (where C is a constant) (1)

(writing a number is too specific ($\frac{1}{2}$ mark).

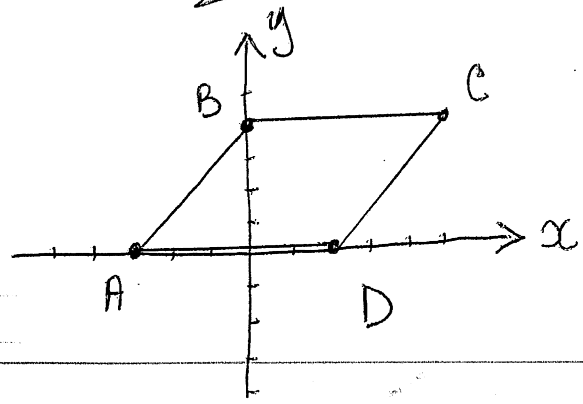
(iv) $\int (3x+1)(x-2)\,dx$

$= \int (3x^2 - 5x - 2)\,dx$

$= \frac{3x^3}{3} - \frac{5x^2}{2} - 2x + C$

$= x^3 - \frac{5x^2}{2} - 2x + C$ (2)

(b)



gradient BC $\frac{4-4}{5-0} = 0$

gradient AD $\frac{0-0}{2-{}^-3} = 0$

BC // AD

gradient AB $\frac{4-0}{0-{}^-3} = \frac{4}{3}$

gradient CD $\frac{0-4}{2-5} = \frac{4}{3}$

AB // CD.

distance BC $= \sqrt{(5-0)^2+(4-4)^2} = 5$

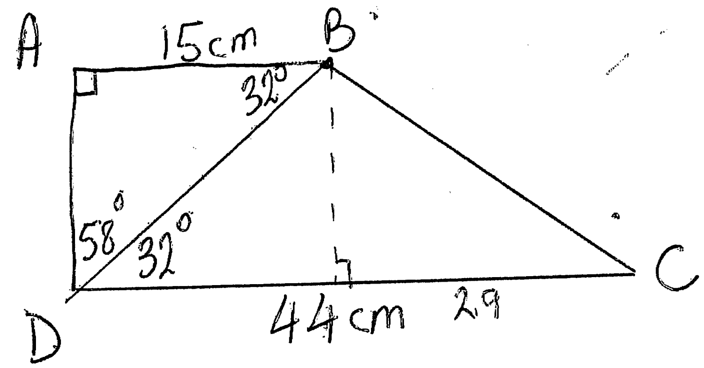
distance AB $= \sqrt{(0-{-3})^2+(4-0)^2} = 5$

So ABCD could be a rhombus. (3)

(ii) midpt AC (1, 2) } diagonals intersect each other.
midpt BD (1, 2)

Gradient AC is $\frac{4-0}{5-3} = \frac{1}{2}$ } since gradients $\times -1$
Gradient BD is $\frac{0-4}{2-0} = -2$ } they are ⊥ (2)

(c)



(i) Is AB // CD ?
Yes, $A\hat{B}D = B\hat{D}C = 32°$
angles in alternate position. (2)

(ii) $\sin 58° = \frac{15}{DB}$

$DB = \frac{15}{\sin 58°} \doteq 17.688$ (3DP)

Now $(BC)^2 = (17.688)^2 + 44^2 - 2 \times 17.688 \times 44 \times \cos 32°$

$BC \doteq 30.477$ 3DP (3)

2 (a) $L = r\theta$

$L = 40 \times \frac{25\pi}{180}$

$\doteqdot 17.5$ cm

$180^\circ = \pi$

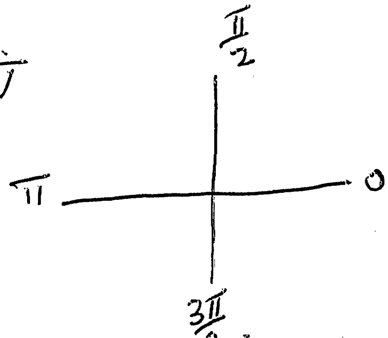
$1^\circ = \frac{\pi}{180}$

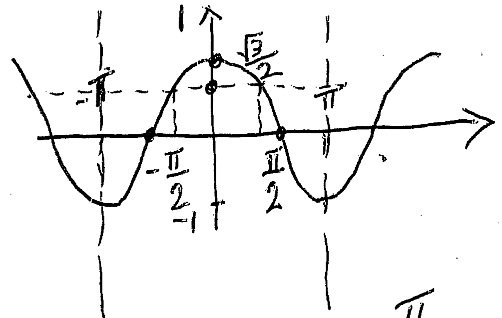
$25^\circ = \frac{25\pi}{180}$ (2)

(b) (i) $\sin\frac{\pi}{4} = \sin 45^\circ = \frac{1}{\sqrt{2}}$ (1)

(ii) $\cot\frac{4\pi}{3} = \cot 240 = \frac{1}{\tan 240^\circ} = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}}$ (1)

(iii) $\cos\theta = \frac{\sqrt{3}}{2}$, $-\pi \leq \theta \leq \pi$





$\theta = \frac{\pi}{6}$ and $-\frac{\pi}{6}$ (2)

(c) (i) $\cos(3x+\pi) \times 3 = 3\cos(3x+\pi)$ (1)

(ii) $\frac{1}{x^2-x+2} \times 2x-1 = \frac{2x-1}{x^2-x+2}$ (1)

(iii) $2e^{2x} - e^{x} + e^{-x}$ (1)

(iv) $\frac{1}{\tan\theta} \times \sec^2\theta = \frac{\sec^2\theta}{\tan\theta}$ or $\frac{\frac{1}{\cos^2\theta}}{\frac{\sin\theta}{\cos\theta}} = \frac{1}{\cos^2\theta} \times \frac{\cos\theta}{\sin\theta}$

$= \frac{1}{\cos\theta\sin\theta}$ (2)

(d) (i) $\sin 2 \doteqdot 0.91$ (1)

(ii) $\ln 17 \doteqdot 2.83$ (1)

(e) let $y = e^{x^2}$

$y' = 2xe^{x^2}$

$y'' = 2x \cdot 2xe^{x^2} + 2e^{x^2} = 4x^2e^{x^2} + 2e^{x^2}$ (2)

Section B

Question 3

a) i $\int_0^{\pi/4} \cos 2x \, dx = 0.5$

$$LHS = \int_0^{\pi/4} \cos 2x \, dx$$

$$= \left[\frac{1}{2}\sin 2x\right]_0^{\pi/4}$$

$$= \frac{1}{2}\sin\frac{2\pi}{4} - \frac{1}{2}\sin 0$$

$$= \tfrac{1}{2} - 0$$

$$= \tfrac{1}{2} = RHS.$$

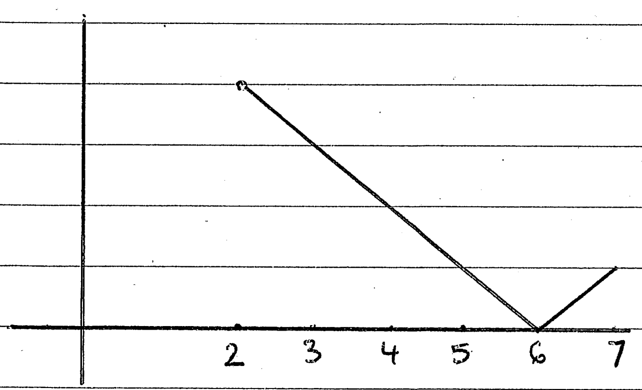
(2)

ii $\int_2^7 \left|3 - \frac{x}{2}\right| dx = 4.25$

x	2	3	4	5	6	7
$f(x)$	2	$1\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$

$\therefore f(x) = |3 - x/2| \quad 2 \leq x \leq 7$

looks like



$\therefore \int_2^7 \left|3 - \frac{x}{2}\right| dx$ = area of big Δ + area of small Δ

$$A = \tfrac{1}{2} \times 4 \times 2 + \tfrac{1}{2} \times 1 \times \tfrac{1}{2}$$

$$= 4.25.$$

OR

$$\int_2^6 3 - \frac{x}{2} \, dx + \int_6^7 -\left(3 - \frac{x}{2}\right) dx$$

$$= \left[3x - \frac{x^2}{4}\right]_2^6 + \left[\frac{x^2}{4} - 3x\right]_6^7$$

$$= \left(18 - \frac{36}{4}\right) - (6 - 1) + \left(\frac{49}{4} - 21\right) - \left(\frac{36}{4} - 18\right)$$

$$= 9 - 5 + -\frac{35}{4} + 9$$

$$= 4.25$$

(3)

b) Circle through (3,0) & (12,0), tangent to y axis.

∴ distances from centre of circle to (3,0) A, (12,0) B or (0,y) C - tangent are equal - radii.

let centre = (x_1, y_1)

Dof A to Centre = Dof B to Centre.

$$\sqrt{(3-x_1)^2+(0-y_1)^2} = \sqrt{(12-x_1)^2+(0-y_1)^2}$$

$$(3-x_1)^2+(-y_1)^2 = (12-x_1)^2+(-y_1)^2$$

$$9-6x_1+x_1^2+y_1^2 = 144-24x_1+x_1^2+y_1^2$$

$$18x_1 = 135$$

$$\therefore x_1 = 7.5$$

DofA to Centre = Dof Tangent to Cent

The y_1 value must be equal to the y value of the tangent.

tangent to circle makes ⊥ with radii.

$$\therefore \sqrt{(3-7.5)^2+(0-y_1)^2} = \sqrt{(7.5-0)^2+(y_1-y)^2}$$

$$\therefore \sqrt{(-4.5)^2+(-y_1)^2} = \sqrt{(7.5)^2+0^2}$$

$$(-4.5)^2+y_1^2 = 7.5^2$$

$$y_1^2 = 7.5^2-(-4.5)^2$$

$$= \pm 6$$

∴ Centre of circle is $(7.5, \pm 6)$

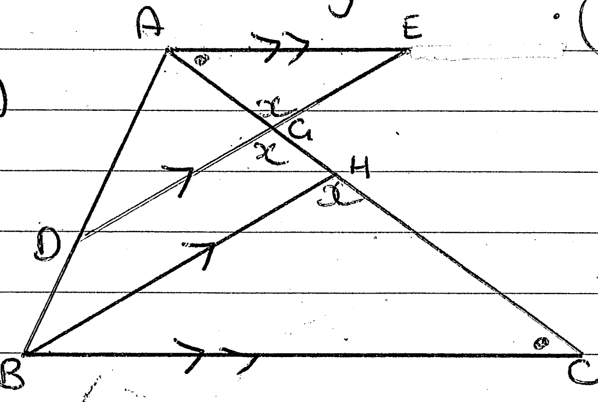
radius = 7.5

∴ eqn of circle

$$(x-7.5)^2+(y\pm 6)^2 = 56.25$$

(4)

c)



i) let ∠AGE = x (3)

∴ ∠DGH = x (vert opp)

∴ ∠BHC = x (corresponding)

(3)

ii) ∠AEG = ∠ACB (alt ∠'s)

∠BHC = ∠AGE (as above)

∴ ∠AEG = ∠CBH (∠sum△)

∴ △AEG ||| △CBH

Question 4

a) $\sin x = 0.8$

$\therefore x = \sin^{-1} 0.8$

$0 \leq x \leq 2\pi$

$\therefore x = 0.9273, 2.2143$

b) i $\int_0^{\pi/4} \sec^2 x \, dx$

$= \left[\tan x\right]_0^{\pi/4}$

$= 1 - 0$

$= 1$

ii $\int_0^{\pi/4} \sec^2 x \, dx$

$\doteq \frac{h}{2}\left[y_0 + y_n\right] + 2\left[y_1\right]$

$h = \frac{b-a}{n} = \frac{\pi/4 - 0}{2} = \frac{\pi}{8}$

$\doteq \frac{\pi/8}{2}\left[\sec^2 0 + \sec^2 \frac{\pi}{4}\right] + 2\left[\sec^2 \frac{\pi}{8}\right]$

$= \frac{\pi}{16}\left[1 + 2\right] + 2.343145751$

$= 1.049212421$

iii $\int_0^{\pi/4} \sec^2 x \, dx$

$\doteq \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

$= \frac{\pi/4 - 0}{6}\left[\sec^2 0 + 4\sec^2 \frac{\pi}{8} + \sec^2 \frac{\pi}{4}\right]$

$= \frac{\pi}{24}\left[1 + 4.686291501 + 2\right]$

$= 1.006133205$

iv Simpsons rule is generally more accurate as it uses parabolic arcs instead of straight lines.

c) $V = \pi \int_{-1}^{1} \left[2(e^{0.5x} + e^{-0.5x})\right]^2 dx$

$= \pi \int_{-1}^{1} 4(e^{x} + e^{-x} + 2)\, dx$

$= 4\pi \left[e^{x} + \frac{e^{-x}}{-1} + 2x\right]_{-1}^{1}$

$= 4\pi \left[e^{x} - e^{-x} + 2x\right]_{-1}^{1}$

$= 4\pi \left[\left(e - \frac{1}{e} + 2\right) - \left(\frac{1}{e} - e - 2\right)\right]$

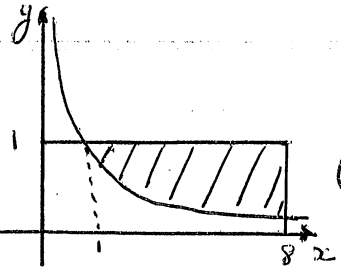
$= 4\pi \left[e - \frac{1}{e} + 2 - \frac{1}{e} + e + 2\right]$

$= 4\pi \left[2e + 4 - \frac{2}{e}\right]$

$= 8\pi\left(e + 2 - \frac{1}{e}\right)$

Question 5:

(a) (i)



(2)

(ii) $\text{Area} = 7 \times 1 - \int_1^8 \frac{1}{x}\,dx$

$= 7 - \left[\ln x\right]_1^8$

$= 7 - \{\ln 8 - \ln 1\}$

$= 7 - \ln 8$ units^2 (2)

$(\approx 4.92055\,8458\ldots)$

(b) $\text{Area} = \int_0^{\pi/3} (\sin 2x - \sin x)\,dx$

$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_0^{\pi/3}$

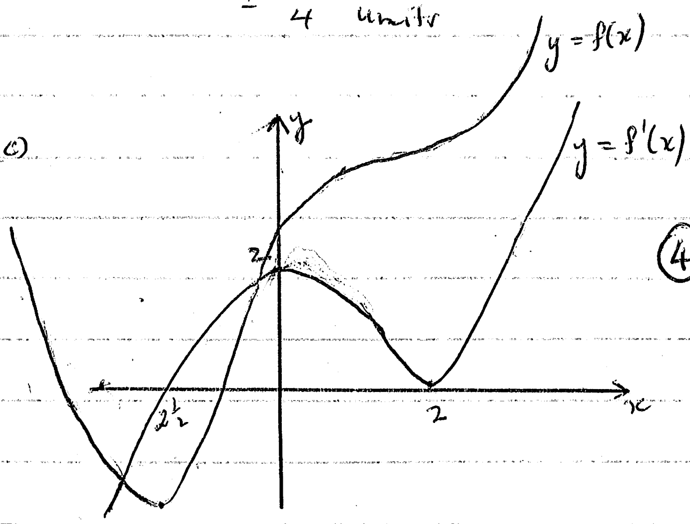
$= \left[-\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right] - \left[-\frac{1}{2}\cos 0 + \cos 0\right]$

$= \left[-\frac{1}{2} \times -\frac{1}{2} + \frac{1}{2}\right] - \left[-\frac{1}{2} + 1\right]$

$= \frac{3}{4} - \frac{1}{2}$ (3)

$= \frac{1}{4}$ units^2

(c)



(4)*

(d) (i) $\frac{d}{dx}(10^x) = \frac{d}{dx}\left(e^{x\ln 10}\right)$

$= \ln 10\, e^{x \ln 10}$

$= \ln 10 .\, 10^x$ (2)

(ii) $\int 10^x\,dx = \int e^{x \ln 10}\,dx$

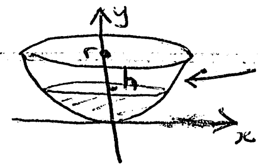
$= \frac{1}{\ln 10} e^{x \ln 10} + c.$

$= \frac{1}{\ln 10} 10^x + c$ (2)

15

Question 6:

(a) (ii)



$x^2 + (y-r)^2 = r^2$

$x^2 = r^2 - (y-r)^2$ (1)

$= 2yr - y^2$

(i) $0 \le h \le r$ (1)

(iii) $\text{Volume} = \pi \int_0^h x^2\,dy$

$= \pi \int_0^h (2yr - y^2)\,dy$

$= \pi \left[y^2 r - \frac{1}{3}y^3\right]_0^h$

$\pi\left[rh^2 - \frac{h^3}{3}\right] - 0$

$= \pi h^2 \left(r - \frac{h}{3}\right)$ (3)

$= \frac{\pi h^2}{3}(3r - h)$

(b) (i) $\frac{d}{dx}(x \ln x) = \ln x . 1 + x . \frac{1}{x}$ (1)

$= \ln x + 1$

(ii) $\therefore \int \frac{d}{dx}(x\ln x)\,dx = \int (\ln x + 1)\,dx$

$\therefore x \ln x = \int \ln x\,dx + x + c.$

$\therefore \int \ln x\,dx = x \ln x - x + c_1$ (2)

(c) $y' = \frac{2}{2x+1}$

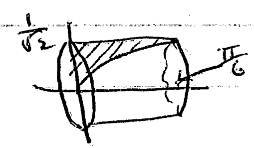
$\therefore y = \ln(2x+1) + c$

When $x = 1$: $\ln 3 = \ln 3 + c$

$\therefore c = 0$ (3)

$\therefore y = \ln(2x+1)$

(d) $\text{Volume} = \pi\left(\frac{1}{\sqrt{2}}\right)^2 \cdot \frac{\pi}{6} - \pi \int_0^{\pi/6} \left(\sqrt{\sin x}\right)^2 dx$



$= \frac{\pi^2}{12} - \pi \int_0^{\pi/6} \sin x\,dx$

$= \frac{\pi^2}{12} + \pi\left[\cos x\right]_0^{\pi/6}$ (4)

$= \frac{\pi^2}{12} + \pi\left[\frac{\sqrt{3}}{2} - 1\right]$ units^3

15