SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

JUNE 2011
ASSESSMENT # 3
YEAR 12

Mathematics

General Instructions:

- Reading time—5 minutes.
- Working time—2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Answer in simplest exact form unless otherwise stated.

Total marks—80 Marks

- Attempt all questions.
- The mark-value of each question is boxed in the right margin.
- Start each NEW section in a separate answer booklet.
- Hand in your answer booklets in 3 sections:
  Section A (Questions 1 and 2),
  Section B (Questions 3 and 4),
  Section C (Questions 5 and 6).

Examiner: Mr P. Bigelow

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Marks

Question 1 (12 marks)

(a) Find a primitive of $3x^2 + 11$. $\boxed{1}$

(b) Express $80°$ in radians. $\boxed{1}$

(c) Approximate the following correct to three decimal places:

(i) $\tan 61°47'$, $\boxed{1}$

(ii) $\cos 3^c$, $\boxed{1}$

(iii) $\dfrac{7}{5e^{1.5}}$. $\boxed{1}$

(d) Find $\displaystyle\int \frac{2x}{7+x^2}dx$. $\boxed{2}$

(e) Express $\sin\dfrac{7\pi}{3}$ in exact form. $\boxed{1}$

(f) Sketch $y = \cos \pi x$ for $|x| \leqslant 1$. $\boxed{2}$

(g) Simplify:

(i) $\ln(e^3)$, $\boxed{1}$

(ii) $e^{4\ln x}$. $\boxed{1}$

Marks

Question 2 (14 marks)

(a) Evaluate:

(i) $\displaystyle\int_0^{\frac{\pi}{3}} 3\sin\frac{x}{2}\,dx$, **2**

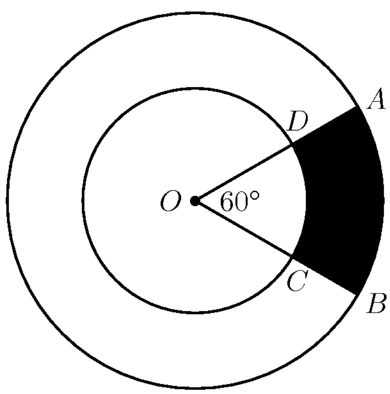
(ii) $\displaystyle\int_0^1 e^{4x}\,dx$. **2**

(b) Differentiate, with respect to x:

(i) $(e^x - 4)^7$, **2**

(ii) $x^2\cos x$. **2**

(c)



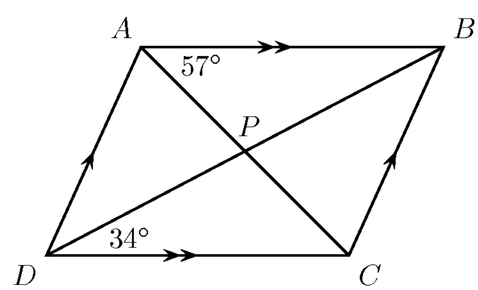
The diagram shows two concentric circles, centre O and radii 15 cm and 25 cm. $A\widehat{O}B = 60°$.

Find correct to three significant figures

(i) the perimeter of the shaded region $ABCD$, **2**

(ii) the area of the shaded region $ABCD$. **2**

(d) **2**



In the diagram $ABCD$ is a parallelogram whose diagonals intersect at P.
Given that $\angle CDB = 34°$ and $\angle BAC = 57°$, find the size of $\angle APB$.
Give reasons for your answer.

Section B

(Use a separate writing booklet.)

Marks

Question 3 (13 marks)

(a) Given the function $y = f(x)$ such that $f'(x) = \dfrac{2}{\sqrt{x}}$ and $f(4) = 1$, find the value of $f(9)$. **2**

(b) Find $\dfrac{d}{dx}\big(\log_e(x-1)\big)$. **1**

(c) Solve the following equations for x:

(i) $3e^{2x} - e^x = 0$, **2**

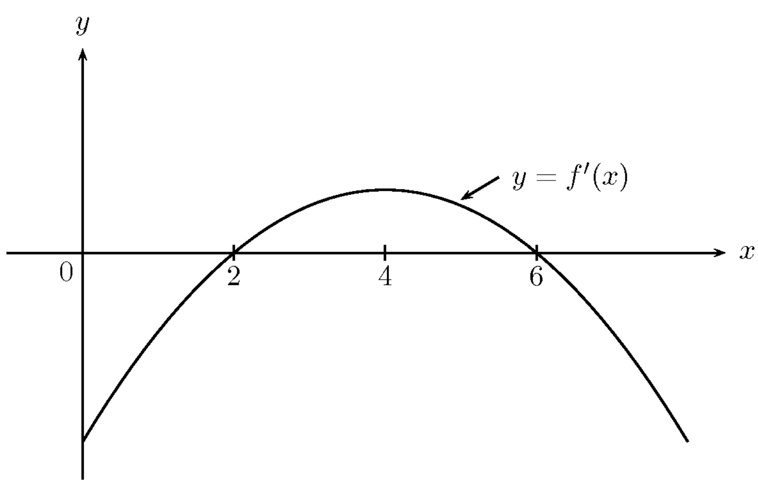
(ii) $\ln(7x - 12) = 2\ln x$. **2**

(d) Given $f(x) = e^{x^3}$, find:

(i) $f'(x)$, **1**

(ii) $f''(x)$. **2**

(e) **3**



The diagram shows the graph of the gradient function of $y = f(x)$.
For what value of x does $y = f(x)$ have a relative maximum?
Justify your answer.

Question 4 (15 marks)

(a) Find the equation of the tangent to $y = e^{x/2}$ at $P(2,\ e)$ and show that it passes through the origin. **3**

(b) Consider the curve $y = 2 + 3x - x^3$.

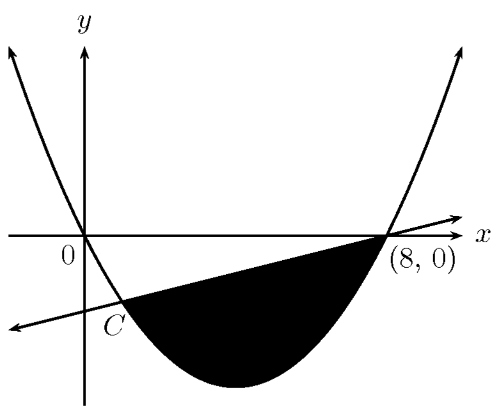
(i) Find $\dfrac{dy}{dx}$. **1**

(ii) Find the stationary points and determine their nature. **3**

(iii) For what values of x is the curve concave up? **2**

(iv) Sketch the curve for $-2 \leqslant x \leqslant 2$. **2**

(c)



The graphs of $y = x - 8$ and $y = x^2 - 8x$ intersect at the points $(8,\ 0)$ and C.

(i) Find the co-ordinates of the point C. **2**

(ii) Find the area of the shaded region bounded by $y = x - 8$ and $y = x^2 - 8x$. **2**

Section C
(Use a separate writing booklet.)

Marks

Question 5 (12 marks)

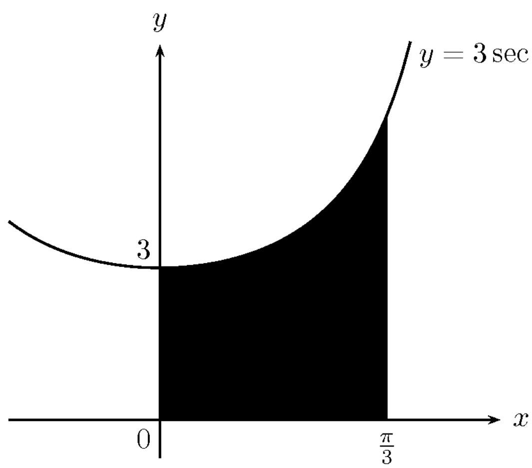
(a) For the points $A(2,\ 7)$ and $B(4,\ 9)$:

(i) Find the co-ordinates of M, the midpoint of AB. **1**

(ii) Show that the equation of the line through M and perpendicular to AB is $x + y - 11 = 0$. **2**

(iii) The line in (ii) meets the x-axis at P. Perpendiculars from A and B to the x-axis meet the x-axis at U and V respectively. Prove that the triangles PAU and BPV are congruent. **3**

(b)



In the diagram, the shaded region is bounded by the curve $y = 3\sec x$, the co-ordinate axes, and the line $x = \frac{\pi}{3}$. **3**

The shaded region is rotated about the x-axis.

Calculate the exact volume of the solid.

(c) Use calculus to prove that for all rectangles with a perimeter of 36 cm, the maximum area is a square. **3**

Marks

Question 6 (14 marks)

(a) (i) Copy and complete the table below with decimals expressed correct to three decimal places. **2**

x	3	4	5	6
$\ln x$				

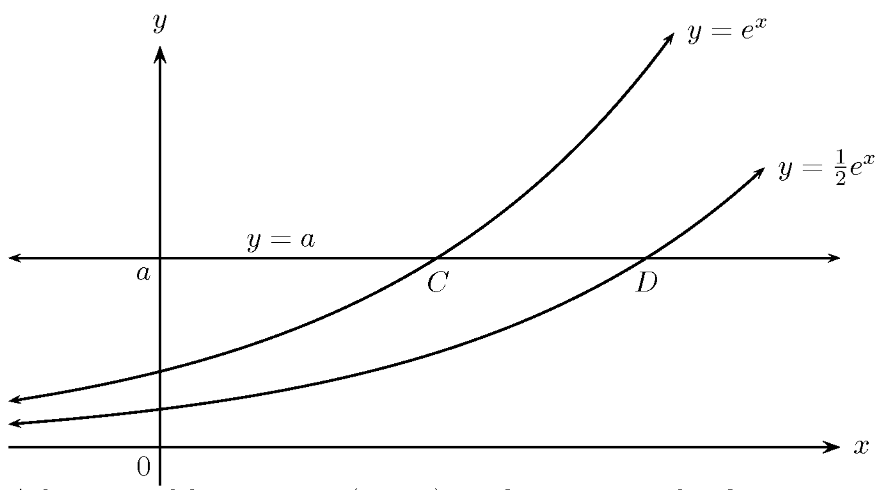
(ii) Using the table and the Trapezoidal Rule, find an approximation to $\int_3^6 \ln x\, dx$. **2**

(iii) Sketch a graph of $y = \ln x$ and use it to explain why the approximation in (ii) will be less than the exact value of the integral. **2**

(iv) Show that $\frac{d}{dx}\big(x \ln x - x\big) = \ln x$. **2**

(v) Deduce the value of the integral in (ii) correct to three decimal places. **2**

(b)



A horizontal line, $y = a$ $(a > 1)$, is drawn as in the diagram cutting $y = e^x$ and $y = \frac{1}{2}e^x$ at C and D respectively.

(i) Find the points C and D. **2**

(ii) Show that CD is a constant length. **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

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Mathematics Solutions

Section A

Marks

Question 1 (12 marks)

(a) Find a primitive of $3x^2 + 11$. 1

Solution: $x^3 + 11x + c$.

(b) Express 80° in radians. 1

Solution: $80^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{4\pi}{9}$.

(c) Approximate the following correct to three decimal places:

(i) $\tan 61^\circ 47'$, 1

Solution: 1.864

(ii) $\cos 3^c$, 1

Solution: -0.990

(iii) $\dfrac{7}{5e^{1.5}}$. 1

Solution: 0.312

(d) Find $\displaystyle\int \frac{2x}{7 + x^2}dx$. 2

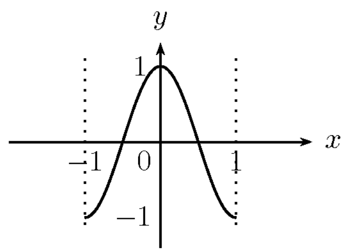
Solution: $\ln(7 + x^2) + c$.

(e) Express $\sin \dfrac{7\pi}{3}$ in exact form. 1

Solution:
$$\begin{aligned} \sin \frac{7\pi}{3} &= \sin \frac{\pi}{3}, \\ &= \frac{\sqrt{3}}{2}. \end{aligned}$$

(f) Sketch $y = \cos \pi x$ for $|x| \leqslant 1$. [2]

Solution:



(g) Simplify:

(i) $\ln(e^3)$, [1]

Solution: 3

(ii) $e^{4 \ln x}$. [1]

Solution: x^4

Marks

Question 2 (14 marks)

(a) Evaluate:

(i) $\displaystyle\int_0^{\frac{\pi}{3}} 3\sin\frac{x}{2}\,dx,$ [2]

Solution: $3\left[-2\cos\frac{x}{2}\right]_0^{\frac{\pi}{3}} = -6\left\{\cos\frac{\pi}{6} - \cos 0\right\},$
$= -6\left\{\frac{\sqrt{3}}{2} - 1\right\},$
$= 6 - 3\sqrt{3}.$

(ii) $\displaystyle\int_0^1 e^{4x}dx.$ [2]

Solution: $\left[\frac{1}{4}e^{4x}\right]_0^1 = \frac{1}{4}(e^4 - 1).$

(b) Differentiate, with respect to x:

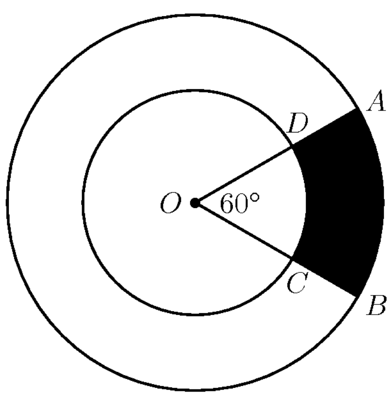
(i) $(e^x - 4)^7,$ [2]

Solution: $7e^x(e^x - 4)^6.$

(ii) $x^2\cos x.$ [2]

Solution: $2x\cos x - x^2\sin x.$

(c)



The diagram shows two concentric circles, centre O and radii 15 cm and 25 cm. $A\widehat{O}B = 60°$.

Find correct to three significant figures

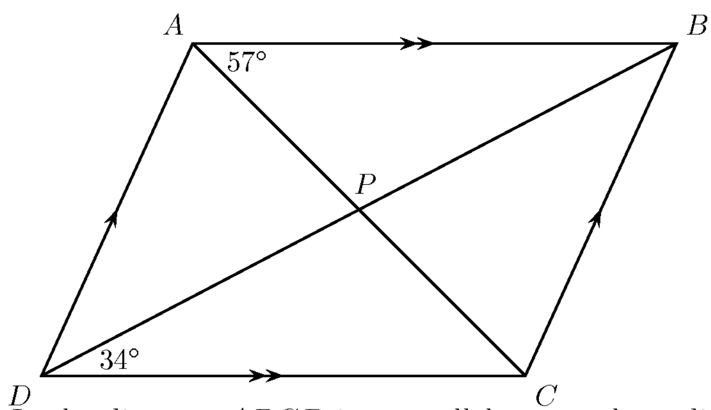
(i) the perimeter of the shaded region $ABCD$, [2]

Solution:
$$\begin{aligned} AB &= \tfrac{60}{360} \times 2 \times \pi \times 25, \\ &= \frac{25\pi}{3}. \\ DC &= \tfrac{1}{6} \times 2 \times \pi \times 15, \\ &= 5\pi. \\ AD &= 10, \\ &= BC. \\ \text{Total} &= \frac{40\pi}{3} + 20, \\ &\approx 61.9\,\text{cm}. \end{aligned}$$

(ii) the area of the shaded region $ABCD$. 2

Solution:
$$\begin{aligned} \text{Sector } AOB &= \tfrac{1}{6} \times \pi \times 25^2 \text{ or } \tfrac{1}{2} \times \frac{\pi}{3} \times 25^2, \\ &= \frac{625\pi}{6}. \\ \text{Sector } DOC &= \tfrac{1}{6} \times \pi \times 15^2 \text{ or } \tfrac{1}{2} \times \frac{\pi}{3} \times 15^2, \\ &= \frac{225\pi}{6}. \\ \text{Total} &= \frac{(625 - 225)\pi}{6}, \\ &\approx 209\,\text{cm}^2. \end{aligned}$$

(d) 2



In the diagram $ABCD$ is a parallelogram whose diagonals intersect at P.
Given that $\angle CDB = 34°$ and $\angle BAC = 57°$, find the size of $\angle APB$.
Give reasons for your answer.

Solution:
$$\begin{aligned} \angle CDB &= \angle PBA \text{ (alt. } \angle\text{s, } AB /\!/ DC), \\ 180° &= 57° + 34° + \angle APB \text{ (} \angle \text{ sum of } \triangle APB), \\ \therefore \angle APB &= 89°. \end{aligned}$$

Section B

(Use a separate writing booklet.)

Marks

Question 3 (13 marks)

(a) Given the function $y = f(x)$ such that $f'(x) = \dfrac{2}{\sqrt{x}}$ and $f(4) = 1$, find the value of $f(9)$. [2]

Solution:
$$\begin{aligned} \int 2x^{-\frac{1}{2}} dx &= 2 \times 2x^{\frac{1}{2}} + c, \\ 1 &= 8 + c \text{ when } x = 4, \\ \therefore f(x) &= 4\sqrt{x} - 7, \\ f(9) &= 5. \end{aligned}$$

(b) Find $\dfrac{d}{dx}\big(\log_e(x-1)\big)$. [1]

Solution: $\dfrac{1}{x-1}$

(c) Solve the following equations for x:

(i) $3e^{2x} - e^x = 0$, [2]

Solution:
$$\begin{aligned} e^x(3e^x - 1) &= 0, \\ 3e^x - 1 &= 0, \text{ as } e^x \neq 0, \\ e^x &= 1/3, \\ x &= \ln 1/3. \end{aligned}$$

(ii) $\ln(7x - 12) = 2\ln x$. [2]

Solution:
$$\begin{aligned} 7x - 12 &= x^2, \\ x^2 - 7x + 12 &= 0, \\ (x-3)(x-4) &= 0, \\ \therefore x &= 3, 4. \end{aligned}$$

(d) Given $f(x) = e^{x^3}$,
find:

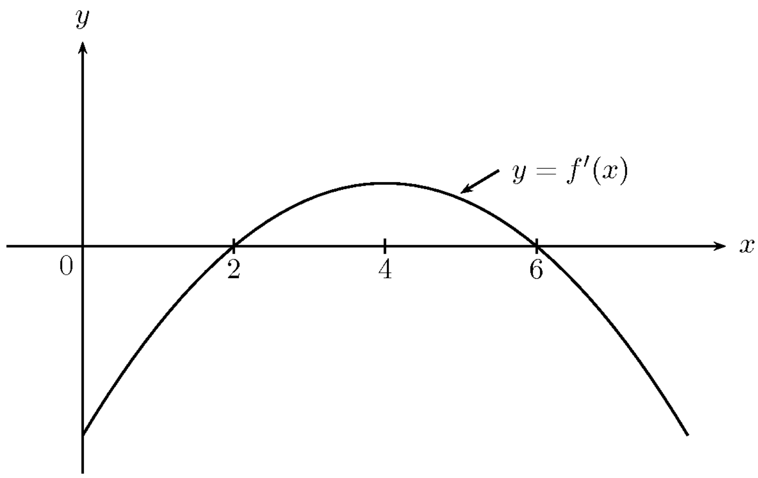
(i) $f'(x)$, [1]

Solution: $3x^2e^{x^3}$

(ii) $f''(x)$. 2

Solution: $3 \times 2x \times e^{x^3} + 3x^2 \times 3x^2 e^{x^3} = e^{x^3}(6x + 9x^4)$.

(e) 3



The diagram shows the graph of the gradient function of $y = f(x)$.
For what value of x does $y = f(x)$ have a relative maximum?
Justify your answer.

Solution: The slope is positive when $2 < x < 6$, zero when $x = 6$, and negative when $x > 6$, so there is a relative maximum at $x = 6$.

Marks

Question 4 (15 marks)

(a) Find the equation of the tangent to $y = e^{x/2}$ at $P(2,\, e)$ and show that it passes through the origin. [3]

Solution:
$$\begin{aligned} \frac{dy}{dx} &= \tfrac{1}{2}e^{x/2}, \\ &= \frac{e}{2} \text{ at } P(2,\, e). \\ \text{Tangent: } y - e &= \frac{e}{2}(x-2), \\ 2y - 2e &= ex - 2e, \\ \textit{i.e.},\ ex - 2y &= 0. \end{aligned}$$
Substituting $(0,\, 0)$ gives
$$\begin{aligned} \text{L.H.S.} &= e \times 0 - 2 \times 0, \\ &= 0, \\ &= \text{R.H.S.} \end{aligned}$$
So the tangent passes through the origin.

(b) Consider the curve $y = 2 + 3x - x^3$.

(i) Find $\dfrac{dy}{dx}$. [1]

Solution: $\dfrac{dy}{dx} = 3 - 3x^2$.

(ii) Find the stationary points and determine their nature. [3]

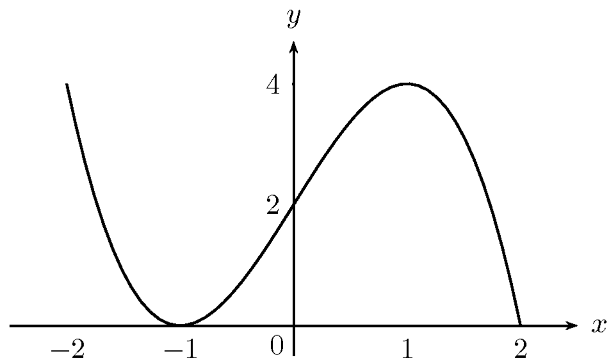
Solution:
$$\begin{aligned} \text{When } 3 - 3x^2 &= 0, \\ x^2 &= 1, \\ x &= \pm 1. \\ \text{If } x &= 1, \quad y = 4, \\ \text{and if } x &= -1, \quad y = 0. \\ \frac{d^2y}{dx^2} &= -6x, \\ &= -6 \text{ when } x = 1, \\ &= 6 \text{ when } x = -1. \end{aligned}$$
So there is a maximum at $(1,\, 4)$ and a minimum at $(-1,\, 0)$.

(iii) For what values of x is the curve concave up? [2]

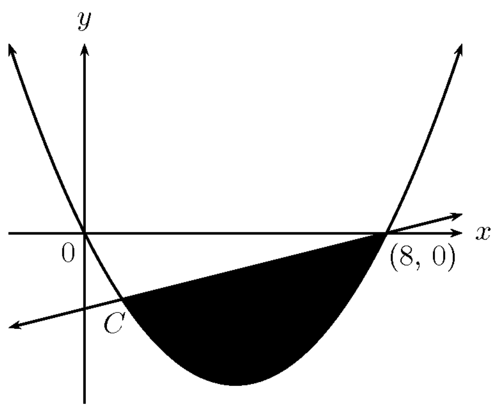
Solution:
$$\begin{aligned} \frac{d^2y}{dx^2} = -6x &> 0, \\ \textit{i.e.},\ x &< 0. \end{aligned}$$

(iv) Sketch the curve for $-2 \leqslant x \leqslant 2$. **2**

Solution:



(c)



The graphs of $y = x - 8$ and $y = x^2 - 8x$ intersect at the points (8, 0) and C.

(i) Find the co-ordinates of the point C. **2**

Solution:
$$\begin{aligned} x^2 - 8x &= x - 8, \\ x(x-8) - (x-8) &= 0, \\ (x-1)(x-8) &= 0, \\ \therefore x &= 1,\ 8, \\ y &= -7,\ 0. \end{aligned}$$
So C is the point $(1, -7)$.

(ii) Find the area of the shaded region bounded by $y = x - 8$ and $y = x^2 - 8x$. **2**

Solution:
$$\begin{aligned} \text{Area} &= \int_1^8 \big((x-8) - (x^2 - 8x)\big)\, dx, \\ &= \int_1^8 (-x^2 + 9x - 8)\, dx, \\ &= \left[-\frac{x^3}{3} + \frac{9x^2}{2} - 8x\right]_1^8, \\ &= -\frac{512}{3} + \frac{576}{2} - 64 - \left(-\frac{1}{3} + \frac{9}{2} - 8\right), \\ &= \frac{343}{6}. \end{aligned}$$

Section C

(Use a separate writing booklet.)

Marks

Question 5 (12 marks)

(a) For the points $A(2,\ 7)$ and $B(4,\ 9)$:

(i) Find the co-ordinates of M, the midpoint of AB. **1**

Solution: $\left(\dfrac{2+4}{2},\ \dfrac{7+9}{2}\right) = (3,\ 8)$

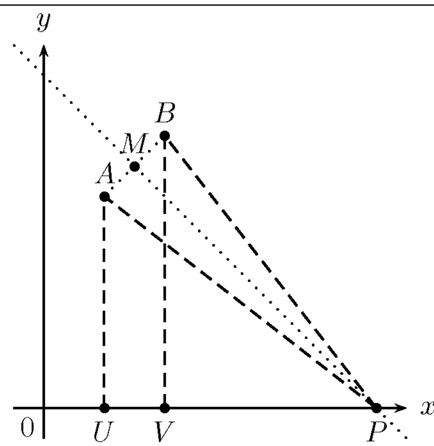
(ii) Show that the equation of the line through M and perpendicular to AB is $x + y - 11 = 0$. **2**

Solution:

$$\begin{aligned} \text{Slope of } AB &= \frac{9-7}{4-2} = 1. \\ y - 8 &= -1 \times (x - 3), \\ y - 8 &= -x + 3, \\ x + y - 11 &= 0. \end{aligned}$$

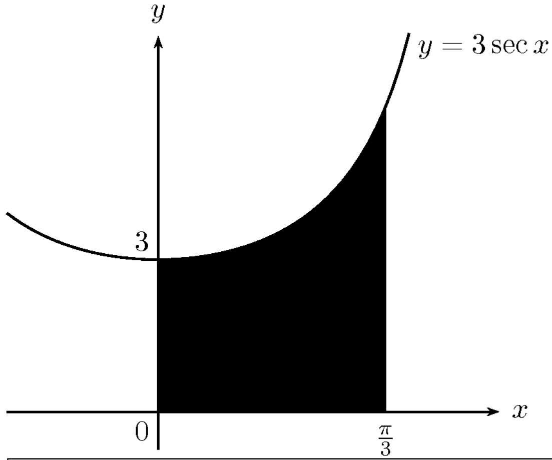
(iii) The line in (ii) meets the x-axis at P. Perpendiculars from A and B to the x-axis meet the x-axis at U and V respectively. Prove that the triangles PAU and BPV are congruent. **3**

Solution:



$$\begin{aligned} \angle PUA &= \angle PVB = 90^\circ\ (AU \perp 0x,\ BV \perp 0x) \\ AU &= 7 - 0 = 7, \\ PV &= 11 - 4 = 7, \\ \therefore AU &= PV, \\ PU &= 11 - 2 = 9, \\ BV &= 9 - 0 = 9, \\ \therefore PU &= BV, \end{aligned}$$

Hence $\triangle PAU \equiv \triangle BPV$ (SAS).

(b) 

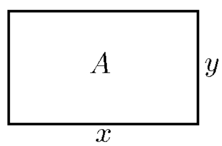
In the diagram, the shaded region is bounded by the curve $y = 3\sec x$, the co-ordinate axes, and the line $x = \frac{\pi}{3}$. **3**

The shaded region is rotated about the x-axis.

Calculate the exact volume of the solid.

Solution:
$$\begin{aligned} \text{Volume} &= \pi \int_0^{\frac{\pi}{3}} 9\sec^2 x\,dx, \\ &= 9\pi\Big[\tan x\Big]_0^{\frac{\pi}{3}}, \\ &= 9\sqrt{3}\,\pi. \end{aligned}$$

(c) Use calculus to prove that for all rectangles with a perimeter of 36 cm, the maximum area is a square. **3**

Solution: 

$$\begin{aligned} 2x + 2y &= 36, \\ y &= 18 - x. \\ \text{Area, } A &= xy, \\ &= 18x - x^2. \\ \frac{dA}{dx} &= 18 - 2x, \\ &= 0 \text{ when } x = 9. \\ \frac{d^2A}{dx^2} &= -2. \end{aligned}$$

$\therefore$ The maximum area occurs when $x = 9$ cm.
i.e., The shape is a square.

Marks

Question 6 (14 marks)

(a) (i) Copy and complete the table below with decimals expressed correct to three decimal places. **2**

x	3	4	5	6
$\ln x$				

Solution:

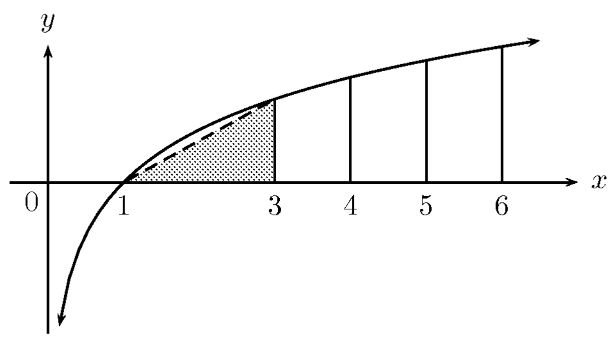
x	3	4	5	6
$\ln x$	1.099	1.386	1.609	1.792

(ii) Using the table and the Trapezoidal Rule, find an approximation to $\int_3^6 \ln x\,dx$. **2**

Solution:
$$\int_3^6 \ln x\,dx \approx \tfrac{1}{2}\{1.099 + 2\times 1.386 + 2\times 1.609 + 1.792\},$$
$$\approx 4.4405 \text{ or about } 4.44.$$

(iii) Sketch a graph of $y = \ln x$ and use it to explain why the approximation in (ii) will be less than the exact value of the integral. **2**

Solution:



As the curve is always concave downwards, any straight line between two points on the curve will always have a small part of the curve area above it (as is shown in exaggerated form between 1 and 3, above). This series of small pieces is not included in the trapeziums between 3 and 4, 4 and 5, and 5 and 6. Therefore the estimated area will be a little less than the real area.

(iv) Show that $\dfrac{d}{dx}\big(x\ln x - x\big) = \ln x$. **2**

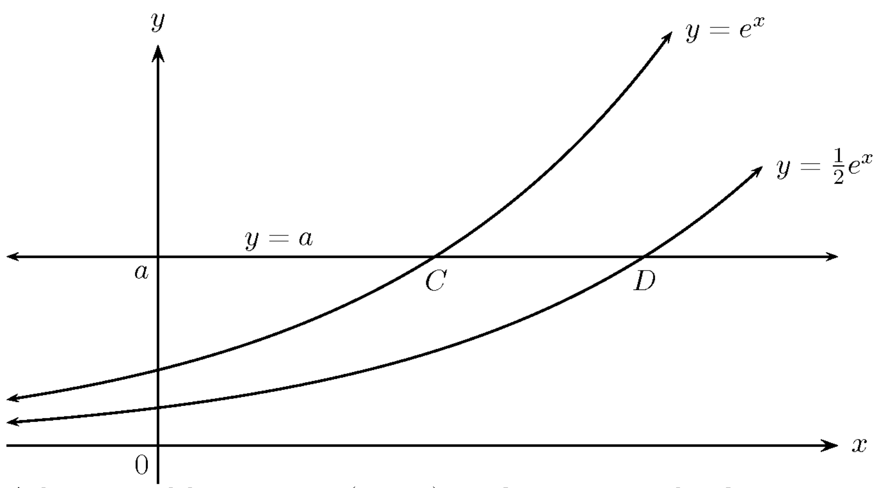
Solution:
$$\begin{aligned}\frac{d}{dx}\big(x\ln x - x\big) &= \ln x + x\times\frac{1}{x} - 1,\\ &= \ln x + 1 - 1,\\ &= \ln x.\end{aligned}$$

(v) Deduce the value of the integral in (ii) correct to three decimal places. 2

Solution: From (iv) we see that

$$\begin{aligned}\int_3^6 \ln x\,dx &= \big[x\ln x - x\big]_3^6,\\ &= 6\ln 6 - 6 - (3\ln 3 - 3),\\ &\approx 4.455 \text{ to three decimal places.}\end{aligned}$$

(b)



A horizontal line, $y = a$ $(a > 1)$, is drawn as in the diagram cutting $y = e^x$ and $y = \frac{1}{2}e^x$ at C and D respectively.

(i) Find the points C and D. 2

Solution:

$$\begin{aligned}\text{At } C,\ a &= e^x,\\ x &= \ln a,\\ \textit{i.e.},\ C &\ \ (\ln a,\, a).\\ \text{At } D,\ a &= \tfrac{1}{2}e^x,\\ x &= \ln 2a,\\ \textit{i.e.},\ D &\ \ (\ln 2a,\, a).\end{aligned}$$

(ii) Show that CD is a constant length. 2

Solution:

$$\begin{aligned}|CD| &= \ln 2a - \ln a,\\ &= \ln\left(\frac{2a}{a}\right),\\ &= \ln 2,\ \text{a constant.}\end{aligned}$$