SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

JUNE 2014
ASSESSMENT # 3
YEAR 12

Mathematics

General Instructions:

- Reading time—5 minutes.
- Working time—2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Answer in simplest exact form unless otherwise stated.

Total marks—80 Marks

- Attempt all questions.
- The mark-value of each question is boxed in the right margin.
- Start each NEW question in a separate answer booklet.
- Hand in your answer booklets in 2 sections:
  Section A (Questions 1, 2, 3),
  Section B (Questions 4, 5, 6).
- Ensure that the graph sheet for Question 4 is inside the booklet for Question 4.

Examiner: Mr E Choy

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section A

Marks

Question 1 (15 marks) (use a separate answer booklet)

(a) Convert 75° to radians (in terms of π). **1**

(b) Differentiate the following with respect to x: **5**

(i) $x^4 + e^2$

(ii) $\sin x^\circ$

(iii) $\ln\left(\dfrac{1}{\sqrt{x+5}}\right)$

(iv) $\tan(\ln x)$

(c) (i) Find the turning points of the curve with equation **5**

$$y = x \ln x.$$

(ii) Find the minimum value of $f(x) = x - \ln x, \quad x > 0$.

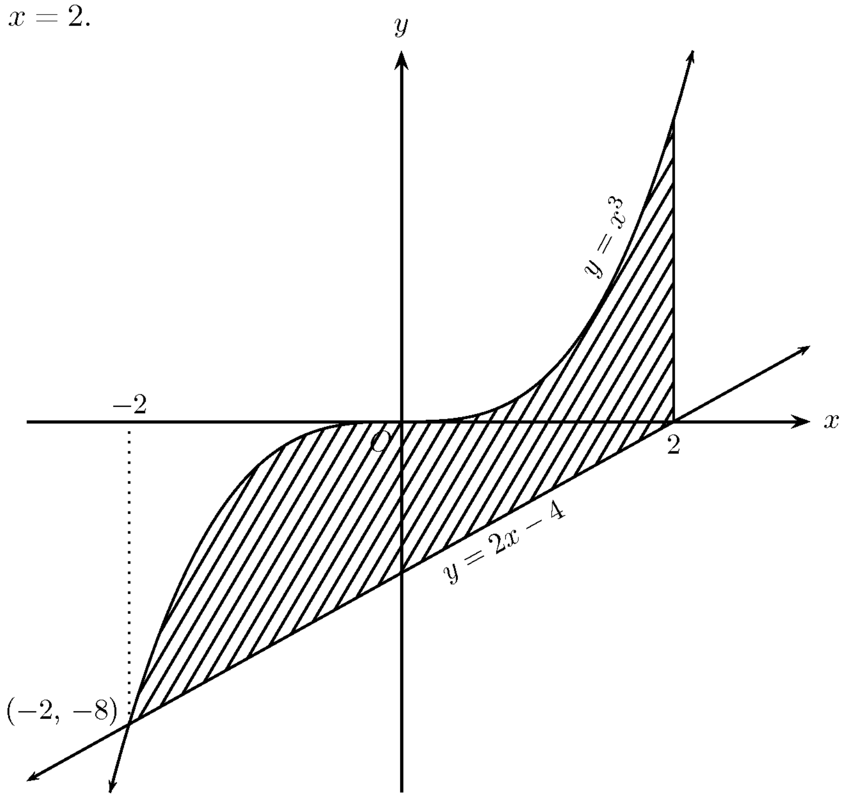
(d) (i) Find any points on the curve $y = x^3 - 6x^2 + 9x + 10$ where the tangent to the curve is parallel to $y = 24x + 12$. **4**

(ii) Suppose ℓ is a tangent to the curve $y = x^3 - 6x^2 + 9x + 10$. What are the possible values that the gradient of ℓ can take?

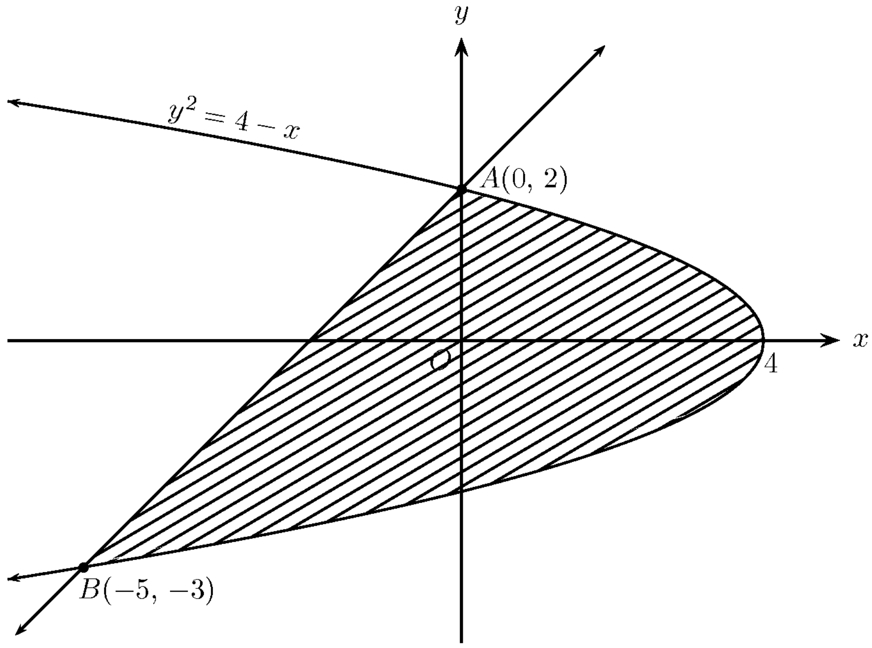
Marks

Question 2 (15 marks) (use a separate answer booklet)

(a) (i) Find the area of the region bounded by the curve $y = x^3$, the lines $y = 2x - 4$ and $x = 2$. **5**



(ii)

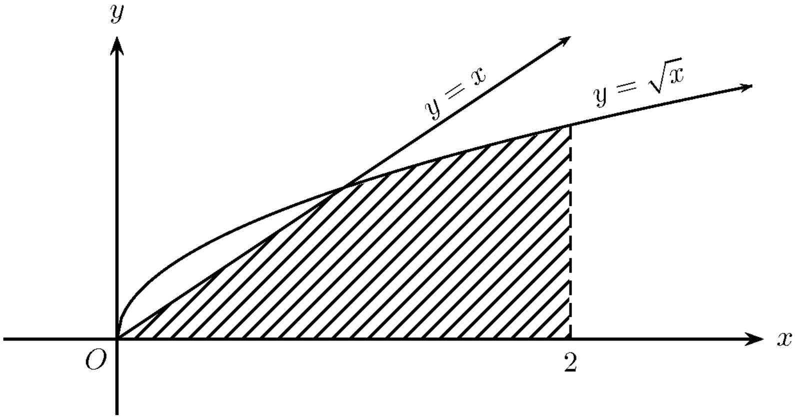


$A(0, 2)$ and $B(-5, -3)$ are two points on the curve $y^2 = 4 - x$.

(α) Find the equation of AB.

(β) Hence find the area of the region bounded by the curve $y^2 = 4 - x$ and the line AB.

(b) (i) Find the volume of the solid generated by revolving the shaded region about the x-axis. 5



(ii) The parabola $y = ax^2 + bx + c$ passes through the points $(-1,\, 2)$, $(1,\, 3)$, and $(3,\, 9)$.

Evaluate $\displaystyle\int_{-1}^{3} \left(ax^2 + bx + c\right) dx$.

(c) AB is the diameter of a semicircle and the chord AP makes an angle of θ radians with AB. If AP divides the semicircle into two equal areas, prove that 3

$$2\theta + \sin 2\theta = \frac{\pi}{2}.$$

(d) $A(a,\, 0)$ and $Q(q,\, 0)$ are points on the positive x-axis, and $B(0,\, b)$ and $P(0,\, p)$ are points on the positive y-axis. Show that 2

$$AB^2 - AP^2 = QB^2 - QP^2.$$

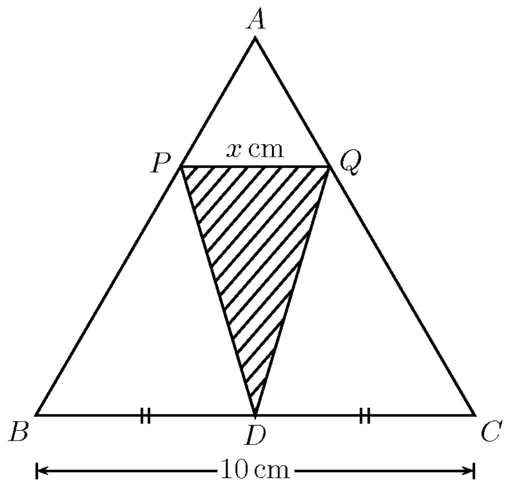
Marks

Question 3 (12 marks) (use a separate answer booklet)

(a) Let $f(x) = \sqrt{1 - x^2}$. Find the interval on which f increases and decreases. $\boxed{3}$

(b) For the function $f(x) = x^3 - 6x^2 + 9x + 1$, find the stationary points and their nature. Find the coördinates of any points of inflexion. Hence sketch the graph. $\boxed{5}$

(c) $\boxed{4}$



$\triangle ABC$ is an equilateral triangle. D is the mid-point of BC and $PQ \parallel BC$, where P and Q are points on AB and AC respectively. Let $PQ = x$ cm.

(i) Prove that the perpendicular distance of P from BC is $\sqrt{3}\left(5 - \dfrac{x}{2}\right)$ cm.

(ii) Express the shaded area, $S\,\text{cm}^2$, in terms of x only.

(iii) Hence find the value of x such that the shaded area is a maximum and find the greatest value of S.

End of Section A

Section B

Marks

Question 4 (17 marks) (use a separate answer booklet)

(a)

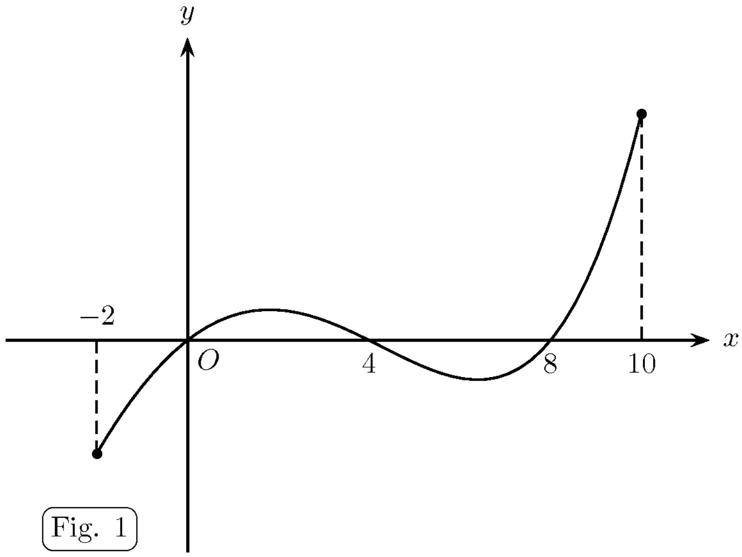


Fig. 1

Let $f(x)$ be a polynomial function where $-2 \leqslant x \leqslant 10$. Fig. 1 shows a sketch of $y = f'(x)$, where $f'(x)$ denotes the first derivative of $f(x)$.

(a) (i) Find the x-coördinates of the maximum and minimum turning points of the curve $y = f(x)$. **9**

(ii) On Fig. 2 of the graph sheet provided, draw a possible sketch of the curve $y = f(x)$.

(iii) Also, on Fig. 3 of the graph sheet provided, sketch the curve $y = f''(x)$.

(iv) Let $g(x) = f(x) + x$, where $-2 \leqslant x \leqslant 10$.
Sketch the curve $y = g'(x)$ on Fig. 4 of the graph sheet.

(v) A student makes the following note:
Since the functions $f(x)$ and $g(x)$ are different, the graphs of $y = f''(x)$ and $y = g''(x)$ should be different.
Explain whether the student is correct or not.

(b) Consider the function $y = f(x)$ where $f(x) = x^2e^x$. **8**
Find

(i) the x- and y-intercepts (if any),

(ii) the stationary points and their nature,

(iii) the points of inflexion.

Hence sketch the graph showing all essential features.

Marks

Question 5 (13 marks) (use a separate answer booklet)

(a) If $f(x) = \dfrac{e^x + e^{-x}}{2}$ and **9**

$g(x) = \dfrac{e^x - e^{-x}}{2}$,

show that

(i) $\big[f(x)\big]^2 - \big[g(x)\big]^2 = 1$.

(ii) $\big[f(x) + g(x)\big]^n = f(nx) + g(nx)$.

(iii) $f(2x) = \big[f(x)\big]^2 + \big[g(x)\big]^2$.

(iv) $f(x) = \sec\theta$ if
$g(x) = \tan\theta$.

(b) The slope at any point $(x,\, y)$ on the curve $\mathcal{C}$ is given by $\dfrac{dy}{dx} = (1-2x)(x+3)$. **4**
$\mathcal{C}$ passes through the point $(0,\, 1)$.

(i) Find the equation of $\mathcal{C}$.

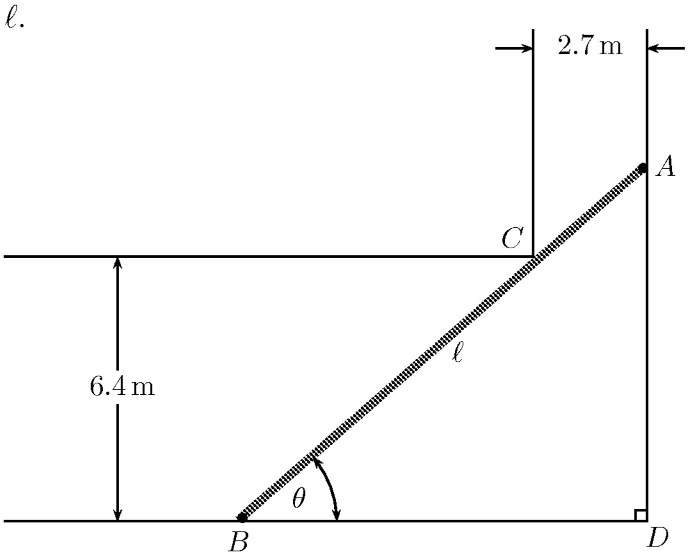
(ii) Find the equation of the normal to $\mathcal{C}$ at the point where $\mathcal{C}$ cuts the y-axis.

Marks

Question 6 (8 marks) (use a separate answer booklet)

Two corridors of a tunnel of width 2.7 m and 6.4 m meet at right-angles as shown below.

A and B are points on the outer wall such that ACB is a straight line. $\angle ABD = \theta$ and $AB = \ell$.



(a) Express ℓ in terms of $\sin\theta$ and $\cos\theta$. **2**

(b) Show that ℓ is minimum when $\tan\theta = \dfrac{4}{3}$. Hence find the minimum value of ℓ. **3**

(c) A rod of length 13 m is moved from one corridor to the other and the height of the ceiling is 4 m. Can the rod be carried round the corner by **3**

(i) pulling the rod on the floor?

(ii) lifting one end of the rod while the other end remains on the floor?

Explain your assertion.

End of Section B

End of Paper

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JUNE 2014
ASSESSMENT # 3
YEAR 12

Mathematics Solutions

Section A

Marks

Question 1 (15 marks) (use a separate answer booklet)

(a) Convert 75° to radians (in terms of π). 1

Solution: $75^\circ \times \dfrac{\pi}{180^\circ} = \dfrac{5\pi}{12}$.

(b) Differentiate the following with respect to x: 5

(i) $x^4 + e^2$

Solution: $\dfrac{d}{dx}\left(x^4 + e^2\right) = 4x^3$.

(ii) $\sin x^\circ$

Solution:
$$\begin{aligned}\frac{d}{dx}\left(\sin x^\circ \times \frac{\pi}{180^\circ}\right) &= \frac{\pi}{180}\cos\frac{\pi x}{180},\\ &= \frac{\pi}{180}\cos x^\circ.\end{aligned}$$

(iii) $\ln\left(\dfrac{1}{\sqrt{x+5}}\right)$

Solution:
$$\begin{aligned}\frac{d}{dx}\left(\ln(x+5)^{-\frac{1}{2}}\right) &= \frac{d}{dx}\left(-\frac{1}{2}\ln(x+5)\right),\\ &= -\frac{1}{2(x+5)}.\end{aligned}$$

(iv) $\tan(\ln x)$

Solution:
$$\begin{aligned}\text{Put } y &= \tan u \quad\text{and}\quad & u &= \ln x,\\ \frac{dy}{du} &= \sec^2 u, & \frac{du}{dx} &= \frac{1}{x},\\ \text{now } \frac{dy}{dx} &= \frac{dy}{du}.\frac{du}{dx}, \quad\text{so} & \frac{dy}{dx} &= \frac{1}{x}.\sec^2(\ln x).\end{aligned}$$

(c) (i) Find the turning points of the curve with equation 5

$$y = x\ln x.$$

Solution:
$$\begin{aligned}\frac{dy}{dx} &= \ln x + \frac{x}{x},\\ &= 1 + \ln x,\\ &= 0 \text{ when } x = e^{-1}.\\ \frac{d^2y}{dx^2} &= \frac{1}{x},\\ &= e \text{ when } x = e^{-1}.\end{aligned}$$

$\therefore$ A minimum turning point at $\left(\dfrac{1}{e}, -\dfrac{1}{e}\right)$.

(ii) Find the minimum value of $f(x) = x - \ln x, \quad x > 0$.

Solution:
$$\begin{aligned} f'(x) &= 1 - \frac{1}{x} = 1 - x^{-1}, \\ &= \frac{x-1}{x}, \\ &= 0 \text{ when } x = 1. \\ f''(x) &= +x^{-2}, \\ &= 1 \text{ when } x = 1. \end{aligned}$$
$\therefore$ The minimum value is 1.

(d) (i) Find any points on the curve $y = x^3 - 6x^2 + 9x + 10$ where the tangent to the curve is parallel to $y = 24x + 12$. 4

Solution: The slope of the tangents is 24.
$$\begin{aligned} y' &= 3x^2 - 12x + 9. \\ \text{When } 3x^2 - 12x + 9 &= 24, \\ x^2 - 4x - 5 &= 0, \\ (x-5)(x+1) &= 0, \\ x &= 5,\ -1. \end{aligned}$$
$\therefore$ The points are $(-1, -6)$ and $(5, 30)$.

(ii) Suppose ℓ is a tangent to the curve $y = x^3 - 6x^2 + 9x + 10$. What are the possible values that the gradient of ℓ can take?

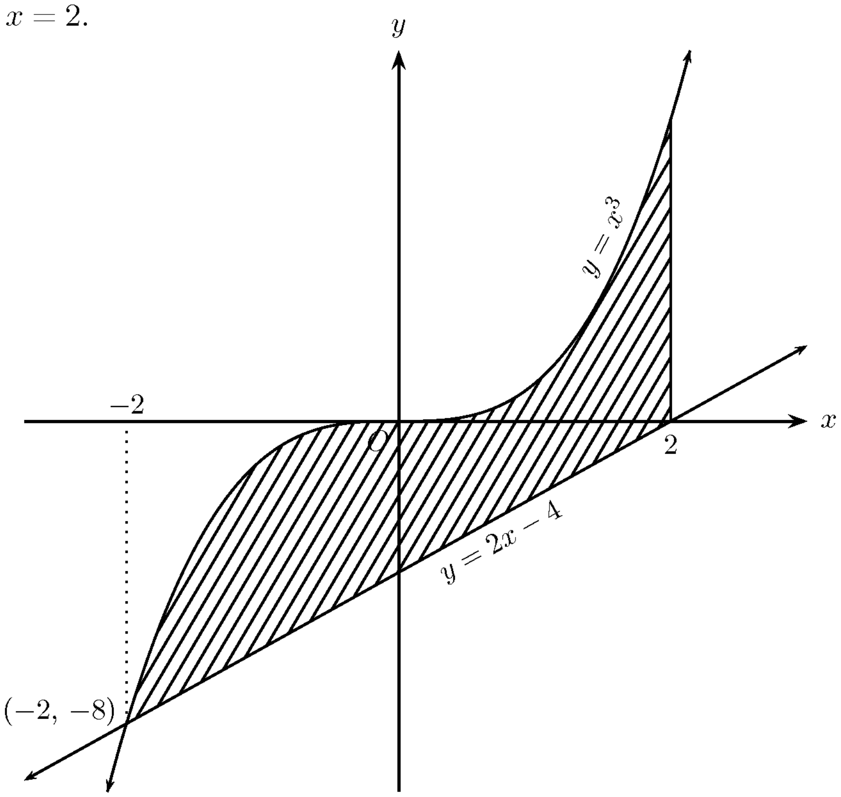
Solution: Method 1—
$$\begin{aligned} \text{Gradient of } \ell &= 3(x^2 - 4x + 3), \\ &= 3(x^2 - 4x + 4 - 1), \\ &= 3(x-2)^2 - 3. \end{aligned}$$
$\therefore$ Gradient of $\ell \geqslant -3$.

Solution: Method 2—
Write the gradient of ℓ as
$$\begin{aligned} y &= 3(x^2 - 4x + 3), \\ y' &= 6x - 12, \\ &= 0 \text{ when } x = 2. \\ y'' &= 6. \end{aligned}$$
$\therefore y$ is a minimum when $x = 2$.
$\therefore$ Gradient of $\ell \geqslant -3$.

Marks

Question 2 (15 marks) (use a separate answer booklet)

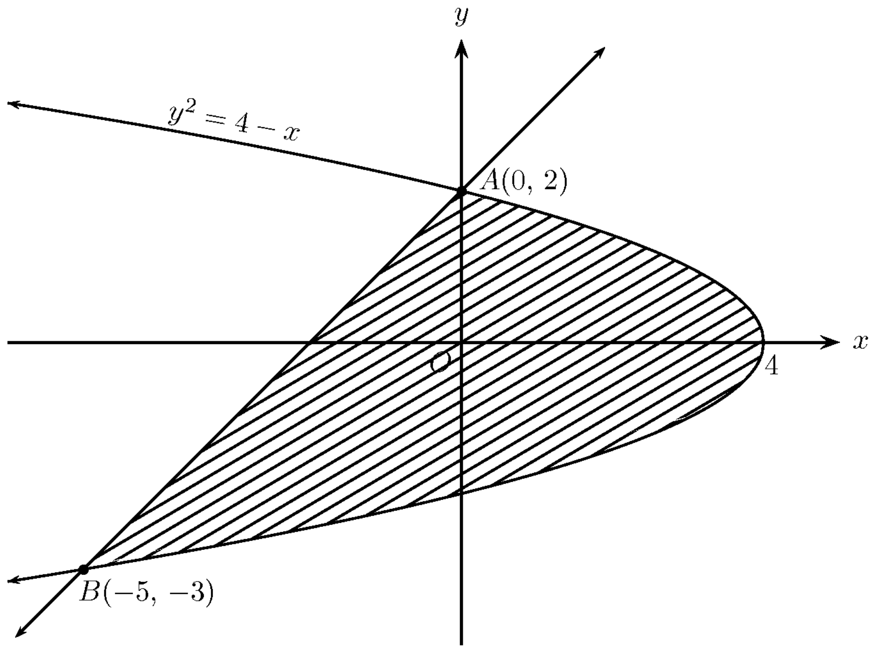
(a) (i) Find the area of the region bounded by the curve $y = x^3$, the lines $y = 2x-4$ and $x = 2$. **5**



Solution:

$$\begin{aligned}\text{Area} &= \int_{-2}^{2} \big(x^3 - (2x-4)\big)dx,\\ &= \int_{-2}^{2} \big(x^3 - 2x + 4\big)dx,\\ &= \left[\frac{x^4}{4} - x^2 + 4x\right]_{-2}^{2},\\ &= (4-4+8) - (4-4-8),\\ &= 16.\end{aligned}$$

(ii)



$A(0,\ 2)$ and $B(-5,\ -3)$ are two points on the curve $y^2 = 4 - x$.

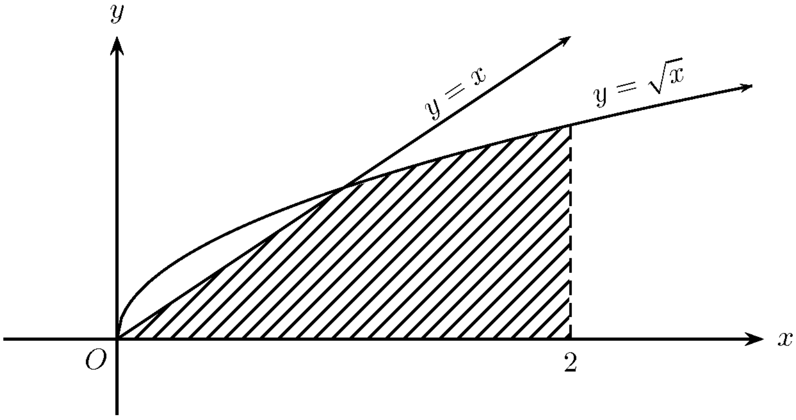
(α) Find the equation of AB.

Solution:
$$\begin{aligned} y - 2 &= \tfrac{-3-2}{-5-0}(x - 0), \\ y - 2 &= x \text{ or } y = x + 2. \end{aligned}$$

(β) Hence find the area of the region bounded by the curve $y^2 = 4 - x$ and the line AB.

Solution: Method 1—
$$\begin{aligned} \text{Area} &= \int_{-3}^{2} \big(4 - y^2 - (y - 2)\big)dy, \\ &= \int_{-3}^{2} \big(6 - y - y^2\big)dy, \\ &= \left[6y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-3}^{2}, \\ &= \big(12 - 2 - \tfrac{8}{3}\big) - \big(-18 - \tfrac{9}{2} + 9\big), \\ &= \frac{125}{6} \text{ or } 20\tfrac{5}{6}. \end{aligned}$$

(b) (i) Find the volume of the solid generated by revolving the shaded region about the x-axis. **5**



Solution: Intersection of the two curves when

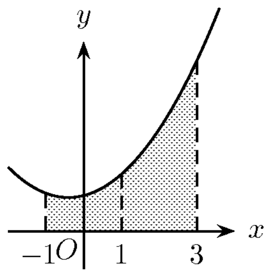
$$\begin{aligned} x &= \sqrt{x}, \\ x^2 &= x, \\ x^2 - x &= 0, \\ x(x-1) &= 0, \\ x &= 0,\ 1 \text{ i.e. at } (0,\ 0) \text{ and } (1,\ 1). \\ \therefore\ \text{Volume} &= \frac{1}{3}.\pi.1^2.1 + \pi\int_1^2 \left(\sqrt{x}\right)^2 dx, \\ &= \pi\left(\frac{1}{3} + \left[\frac{x^2}{2}\right]_1^2\right), \\ &= \frac{\pi}{3} + \pi\left(2 - \tfrac{1}{2}\right), \\ &= \frac{11\pi}{6}. \end{aligned}$$

(ii) The parabola $y = ax^2 + bx + c$ passes through the points $(-1,\ 2)$, $(1,\ 3)$, and $(3,\ 9)$.

Evaluate $\displaystyle\int_{-1}^{3} \left(ax^2 + bx + c\right) dx$.

Solution: Method 1—

$$\begin{aligned} \int_{-1}^{3} \left(ax^2 + bx + c\right) &= \frac{3 - {}^-1}{6}(2 + 4 \times 3 + 9), \\ &= \frac{46}{3} \text{ or } 15\tfrac{1}{3}. \end{aligned}$$



Solution: Method 2—

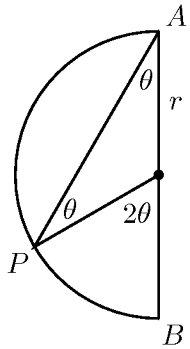
$$\begin{aligned}
y &= ax^2+bx+c \text{ at } (-1,\,2),\ (1,\,3),\ (3,\,9):\\
2 &= a-b+c \ldots\ldots\ldots\ldots \boxed{1}\\
3 &= a+b+c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \boxed{2}\\
9 &= 9a+3b+c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \boxed{3}\\
\boxed{2}-\boxed{1}:\quad 1 &= 2b \implies b=\tfrac{1}{2}.\\
2\tfrac{1}{2} &= a+c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}\\
7\tfrac{1}{2} &= 9a+c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}\\
\text{(B)}-\text{(A)}:\quad 5 &= 8a \implies a=\tfrac{5}{8},\\
\textit{sub. in}\,\text{(A)}:\quad c &= \tfrac{5}{2}-\tfrac{5}{8}=\tfrac{15}{8}.\\
\text{So}\quad y &= \frac{5x^2}{8}+\frac{x}{2}+\frac{15}{8}.\\
\text{Area} &= \frac{1}{8}\int_{-1}^{3}\big(5x^2+4x+15\big)dx,\\
&= \frac{1}{8}\left[\frac{5x^3}{3}+2x^2+15x\right]_{-1}^{3},\\
&= \frac{1}{8}\Big(45+18+45-\big(-\tfrac{5}{3}+2-15\big)\Big),\\
&= \frac{46}{3} \text{ or } 15\tfrac{1}{3}.
\end{aligned}$$

(c) AB is the diameter of a semicircle and the chord AP makes an angle of θ radians with AB. If AP divides the semicircle into two equal areas, prove that 3

$$2\theta+\sin 2\theta=\frac{\pi}{2}.$$

Solution:

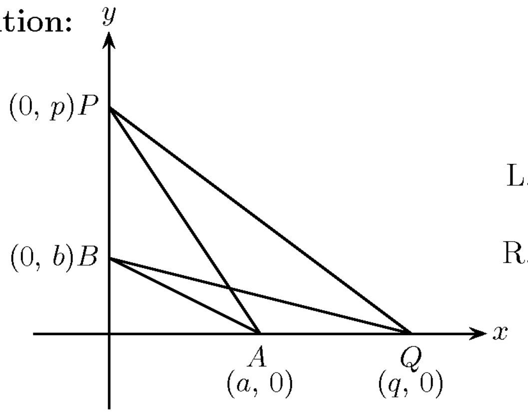
$$\begin{aligned}
\text{Semicircle area} &= \frac{\pi r^2}{2}.\\
\text{Segment area} &= \frac{r^2}{2}\big(\pi-2\theta-\sin(\pi-2\theta)\big).\\
\text{Area } APB &= \frac{r^2}{2}(\pi-\pi+2\theta+\sin 2\theta).\\
\text{For equal areas, } \pi-2\theta-\sin 2\theta &= 2\theta+\sin 2\theta,\\
\pi &= 4\theta+2\sin 2\theta,\\
\textit{i.e.,}\ \tfrac{\pi}{2} &= 2\theta+\sin 2\theta.
\end{aligned}$$



(d) $A(a,\,0)$ and $Q(q,\,0)$ are points on the positive x-axis, and $B(0,\,b)$ and $P(0,\,p)$ are points on the positive y-axis. Show that **2**

$$AB^2 - AP^2 = QB^2 - QP^2.$$

Solution:



$$\begin{aligned} AB^2 &= a^2 + b^2 \\ AP^2 &= a^2 + p^2 \\ QB^2 &= q^2 + b^2 \\ QP^2 &= q^2 + p^2 \\ \text{L.H.S.} &= a^2 + b^2 - \left(a^2 + p^2\right) \\ &= b^2 - p^2 \\ \text{R.H.S.} &= q^2 + b^2 - \left(q^2 + p^2\right) \\ &= b^2 - p^2 \\ &= \text{L.H.S.} \end{aligned}$$

Marks

Question 3 (12 marks) (use a separate answer booklet)

(a) Let $f(x) = \sqrt{1-x^2}$. Find the interval on which f increases and decreases. 3

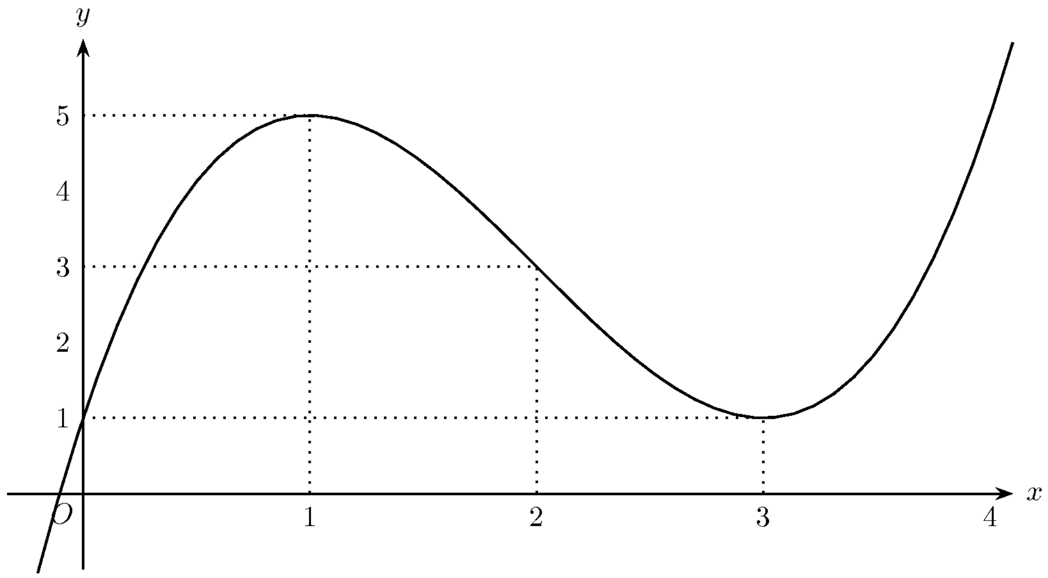
Solution:
$$\begin{aligned} f(x) &= \sqrt{1-x^2}, & -1 \leqslant x \leqslant 1 \\ f'(x) &= \frac{-2x}{2\sqrt{1-x^2}}, & -1 < x < 1 \\ f'(x) &> 0 \text{ when } -1 < x < 0, \\ f'(x) &< 0 \text{ when } \quad 0 < x < 1. \end{aligned}$$
i.e. f increases on the interval $-1 < x < 0$ and decreases on the interval $0 < x < 1$.

(b) For the function $f(x) = x^3 - 6x^2 + 9x + 1$, find the stationary points and their nature. Find the coördinates of any points of inflexion. Hence sketch the graph. 5

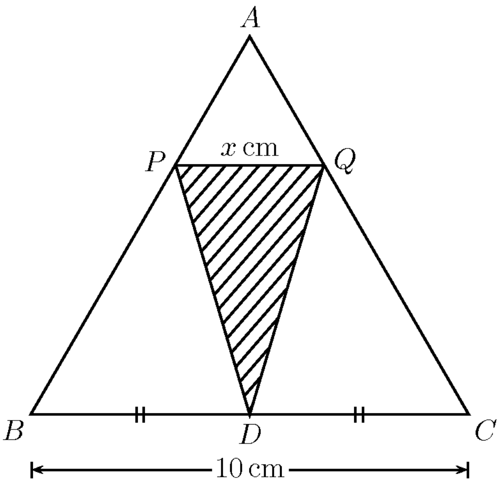
Solution:
$$\begin{aligned} f'(x) &= 3x^2 - 12x + 9, \\ &= 0 \text{ when } x^2 - 4x + 3 = 0, \\ & \quad i.e. \ (x-3)(x-1) = 0, \\ & \quad x = 1,\ 3. \\ f''(x) &= 6x - 12, \\ &= 0 \text{ when } x = 2. \end{aligned}$$

x	1	3
$f''x$	-6	6

$\therefore$ Maximum at (1, 5).
Minimum at (3, 1).
Inflexion at (2, 3).



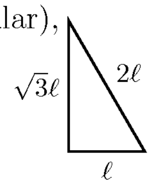
(c) **4**



$\triangle ABC$ is an equilateral triangle. D is the mid-point of BC and $PQ \parallel BC$, where P and Q are points on AB and AC respectively. Let $PQ = x\,\text{cm}$.

(i) Prove that the perpendicular distance of P from BC is $\sqrt{3}\left(5 - \dfrac{x}{2}\right)$ cm.

Solution: $\triangle APQ \;|||\; \triangle ABC$ (equiangular),



$$\begin{aligned} \text{height } \triangle APQ &= \frac{\sqrt{3}x}{2}\,\text{cm}, \\ \text{height } \triangle ABC &= 5\sqrt{3}\,\text{cm}, \\ \text{height difference} &= 5\sqrt{3} - \frac{\sqrt{3}x}{2}\,\text{cm}. \end{aligned}$$

$\therefore$ Perpendicular distance of P from BC is $\sqrt{3}\left(5 - \dfrac{x}{2}\right)$ cm.

(ii) Express the shaded area, $S\,\text{cm}^2$, in terms of x only.

Solution:
$$\begin{aligned} \text{Area } S &= \tfrac{1}{2}x.\sqrt{3}\left(5 - \frac{x}{2}\right), \\ &= \frac{5\sqrt{3}x}{2} - \frac{\sqrt{3}x^2}{4}. \end{aligned}$$

(iii) Hence find the value of x such that the shaded area is a maximum and find the greatest value of S.

Solution:
$$\begin{aligned} \frac{dS}{dx} &= \frac{5\sqrt{3}}{2} - \frac{\sqrt{3}x}{2}, \\ &= 0 \text{ when } x = 5. \\ \frac{d^2S}{dx^2} &= -\frac{\sqrt{3}}{2} \quad \therefore \text{ Maximum when } x = 5. \end{aligned}$$

The greatest value of S is $\dfrac{25\sqrt{3}}{4}\,\text{cm}^2$.

End of Section A

Section B

Marks

Question 4 (17 marks) (use a separate answer booklet)

(a) [7]

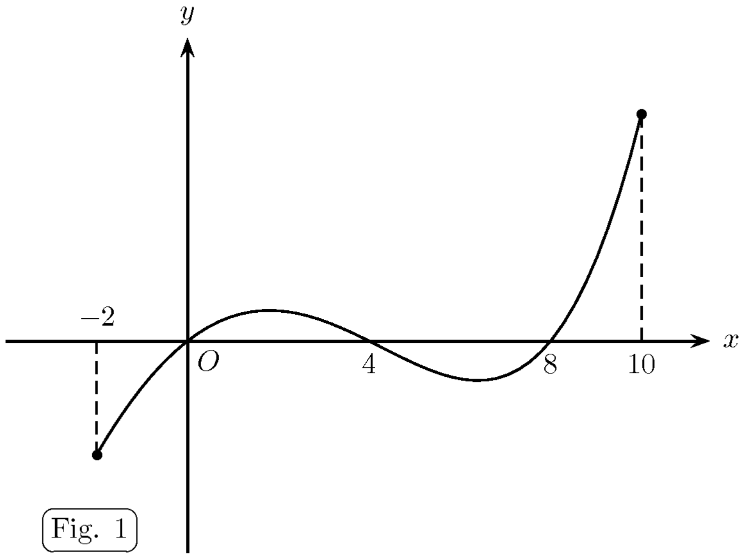


Fig. 1

Let $f(x)$ be a polynomial function where $-2 \leqslant x \leqslant 10$. Fig. 1 shows a sketch of $y = f'(x)$, where $f'(x)$ denotes the first derivative of $f(x)$.

(a) (i) Find the x-coördinates of the maximum and minimum turning points of the curve $y = f(x)$. [6]

Solution: Maximum at $x = 4$.
Minimums at $x = 0, 8$.

(ii) On Fig. 2 of the graph sheet provided, draw a possible sketch of the curve $y = f(x)$.

Solution:

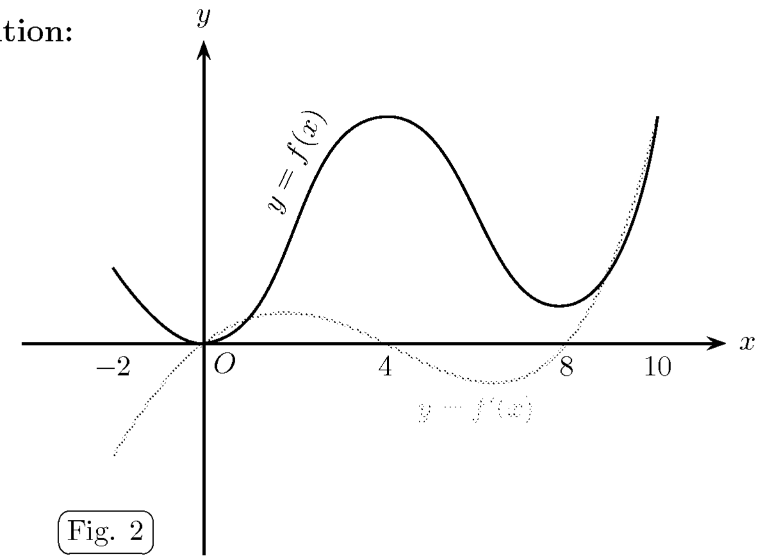
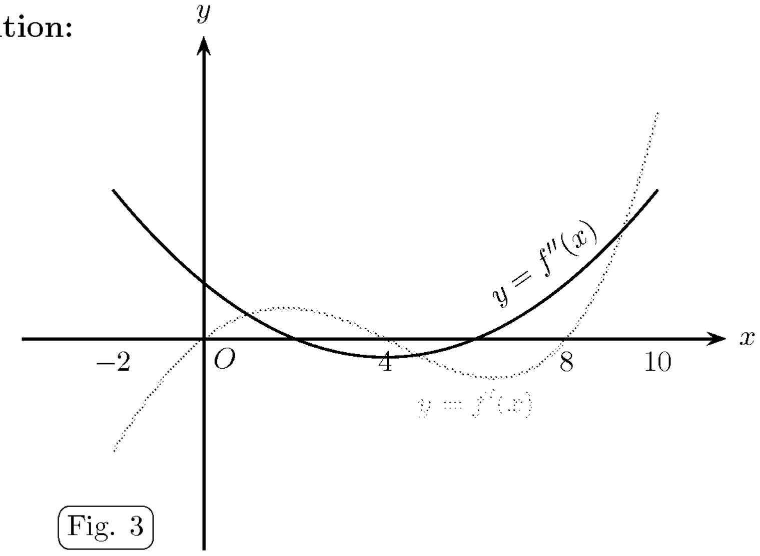


Fig. 2

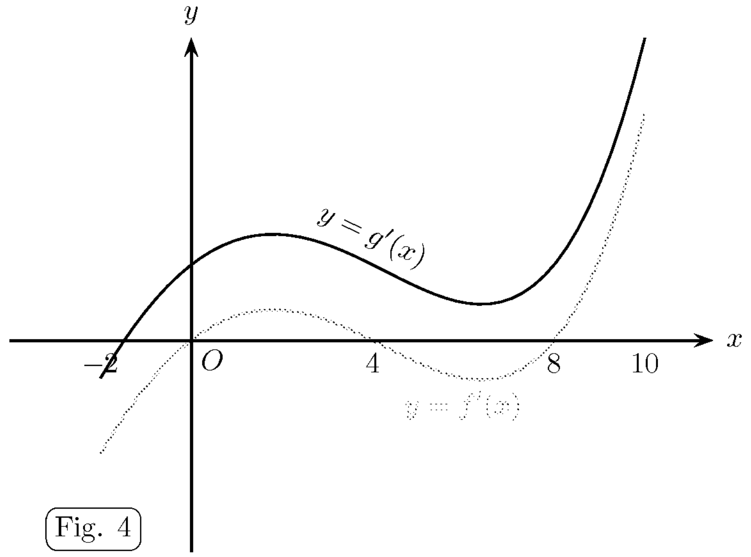
(iii) Also, on Fig. 3 of the graph sheet provided, sketch the curve $y = f''(x)$.

Solution:



(iv) Let $g(x) = f(x) + x$, where $-2 \leqslant x \leqslant 10$.
Sketch the curve $y = g'(x)$ on Fig. 4 of the graph sheet.

Solution: $g'(x) = f'(x) + 1$.



(v) A student makes the following note:

Since the functions $f(x)$ and $g(x)$ are different, the graphs of $y = f''(x)$ and $y = g''(x)$ should be different.

Explain whether the student is correct or not.

Solution: $g''(x) = f''(x) + 0$.
So the student is wrong, as both the second derivatives are equal.

(b) Consider the function $y = f(x)$ where $f(x) = x^2e^x$. **4**

Find

(i) the x- and y-intercepts (if any),

Solution: $x^2 \geqslant 0$ and $e^x > 0$ for all $x \in \mathbb{R}$.
$f(0) = 0$.
$\therefore$ Intercept is at (0, 0) only.

(ii) the stationary points and their nature,

Solution:

$$\begin{aligned} f'(x) &= 2xe^x + x^2e^x, \\ &= xe^x(2+x), \\ &= 0 \text{ when } x = -2,\ 0. \\ f''(x) &= 2e^x + 2xe^x + 2xe^x + x^2e^x, \\ &= e^x\big(x^2+4x+2\big), \\ &= e^x\big(x^2+4x+4-2\big), \\ &= e^x\big((x+2)^2-2\big), \\ &= 0 \text{ when } x = -2 \pm \sqrt{2}. \end{aligned}$$

x	-2	0
$f''x$	-0.27	2

$\therefore$ Maximum at $\left(-2,\ \dfrac{4}{e^2}\right) \approx (-2,\ 0.54)$.
Minimum at (0, 0).

(iii) the points of inflexion.

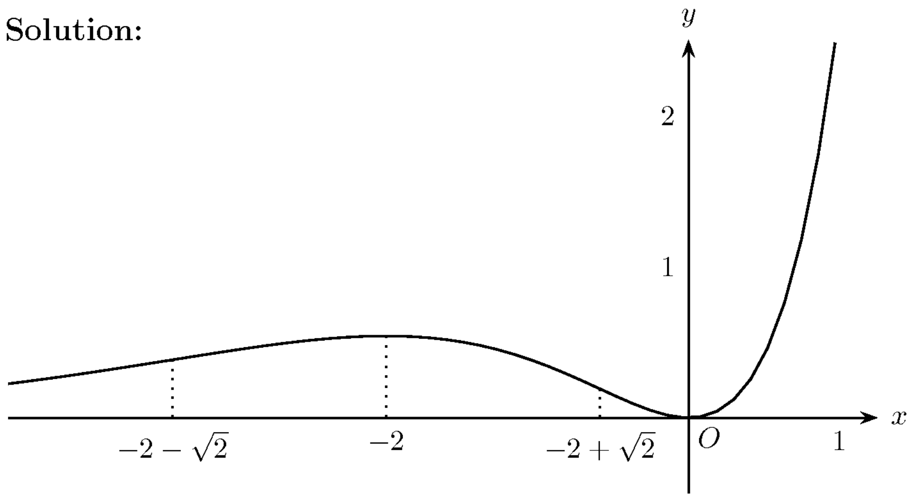
Solution:

x	-4	-3	-1	0
$f''(x)$	0.036	-0.05	-0.37	2

Inflexions at $\big(-2-\sqrt{2},\ (6+4\sqrt{2})e^{-2-\sqrt{2}}\big) \approx (-3.41,\ 0.38)$,
and at $\big(-2+\sqrt{2},\ (6-4\sqrt{2})e^{-2+\sqrt{2}}\big) \approx (-0.58,\ 0.19)$.

Hence sketch the graph showing all essential features.

Solution:



Marks

Question 5 (13 marks) (use a separate answer booklet)

(a) If $f(x) = \dfrac{e^x + e^{-x}}{2}$ and **9**

$g(x) = \dfrac{e^x - e^{-x}}{2}$,

show that

(i) $\big[f(x)\big]^2 - \big[g(x)\big]^2 = 1$.

> **Solution:** $\begin{aligned} \text{L.H.S.} &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}, \\ &= 1, \\ &= \text{R.H.S.} \end{aligned}$

(ii) $\big[f(x) + g(x)\big]^n = f(nx) + g(nx)$.

> **Solution:** $\begin{aligned} f(x) + g(x) &= e^x, \\ \text{L.H.S.} &= \big[f(x) + g(x)\big]^n, \\ &= \big(e^x\big)^n, \\ &= e^{nx}. \\ \text{R.H.S.} &= \frac{e^{nx} + e^{-nx} + e^{nx} - e^{-nx}}{2}, \\ &= e^{nx}, \\ &= \text{L.H.S.} \end{aligned}$

(iii) $f(2x) = \big[f(x)\big]^2 + \big[g(x)\big]^2$.

> **Solution:** $\begin{aligned} \text{L.H.S.} &= \frac{e^{2x} + e^{-2x}}{2}. \\ \text{R.H.S.} &= \frac{e^{2x} + 2 + e^{-2x}}{4} + \frac{e^{2x} - 2 + e^{-2x}}{4}, \\ &= \frac{e^{2x} + e^{-2x}}{2}, \\ &= L.H.S. \end{aligned}$

(iv) $\begin{aligned} f(x) &= \sec\theta \text{ if} \\ g(x) &= \tan\theta. \end{aligned}$

> **Solution:** If $g(x) = \tan\theta$ and $\tan^2\theta + 1 = \sec^2\theta$,
>
> then $\begin{aligned} \sec^2\theta &= \frac{e^{2x} - 2 + e^{-2x}}{4} + \frac{4}{4}, \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4}, \\ &= \big(f(x)\big)^2. \\ \therefore f(x) &= \sec\theta. \end{aligned}$

(b) The slope at any point $(x,\ y)$ on the curve $\mathcal{C}$ is given by $\dfrac{dy}{dx} = (1-2x)(x+3)$. 4

$\mathcal{C}$ passes through the point $(0,\ 1)$.

(i) Find the equation of $\mathcal{C}$.

Solution:
$$y' = -2x^2 - 5x + 3.$$
$$y = -\frac{2x^3}{3} - \frac{5x^2}{2} + 3x + c.$$
Substitute $(0,\ 1)$: $1 = c$,
$\therefore$ Equation of $\mathcal{C}$ is $y = -\dfrac{2x^3}{3} - \dfrac{5x^2}{2} + 3x + 1.$

(ii) Find the equation of the normal to $\mathcal{C}$ at the point where $\mathcal{C}$ cuts the y-axis.

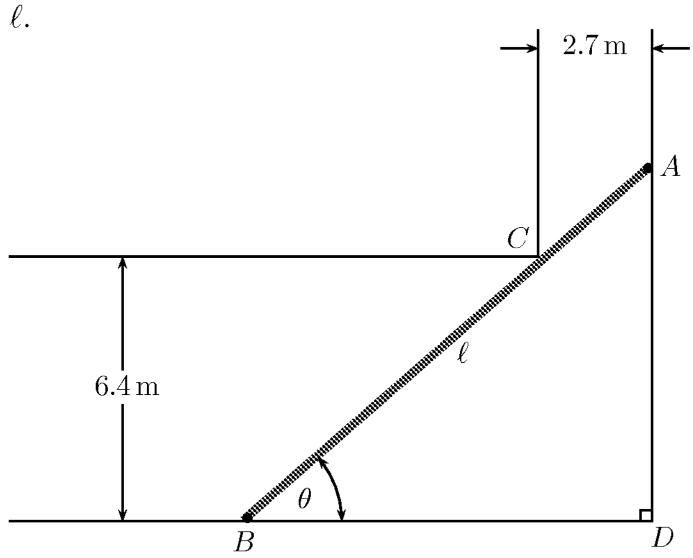
Solution: When $x = 0,\ y = 1,\ y' = 3.$
$\therefore$ Normal equation is$y - 1 = -\frac{1}{3}(x - 0),$
$$3y - 3 = -x,$$
$$x + 3y - 3 = 0.$$

Marks

Question 6 (8 marks) (use a separate answer booklet)

Two corridors of a tunnel of width 2.7 m and 6.4 m meet at right-angles as shown below.

A and B are points on the outer wall such that ACB is a straight line. $\angle ABD = \theta$ and $AB = \ell$.



(a) Express ℓ in terms of $\sin\theta$ and $\cos\theta$. **2**

Solution: $\cos\theta = \dfrac{2.7}{AC}$ $\qquad \sin\theta = \dfrac{6.4}{BC}$

$$\begin{aligned} \ell &= AC + CB, \\ &= \frac{2.7}{\cos\theta} + \frac{6.4}{\sin\theta}, \\ &= 2.7\sec\theta + 6.4\text{cosec}\,\theta. \end{aligned}$$

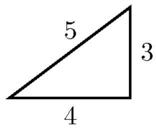
(b) Show that ℓ is minimum when $\tan\theta = \dfrac{4}{3}$. Hence find the minimum value of ℓ. **3**

Solution:

$$\begin{aligned} \frac{d\ell}{d\theta} &= 2.7\sec\theta\tan\theta - 6.4\text{cosec}\,\theta\cot\theta, \\ &= \frac{2.7\sin\theta}{\cos^2\theta} - \frac{6.4\cos\theta}{\sin^2\theta}, \\ &= 0 \text{ when } 2.7\sin^3\theta = 6.4\cos^3\theta, \\ \tan^3\theta &= \frac{64}{27}, \\ \textit{i.e. } \tan\theta &= \frac{4}{3}, \\ \theta &\approx 0.927 \end{aligned}$$

θ	0.9	1
$\dfrac{d\ell}{d\theta}$	-1	2.9

$\therefore$ Maximum $\ell = 2.7 \div \frac{3}{5} + 6.4 \div \frac{4}{5}$



$= 12\frac{1}{2}$ m.

(c) A rod of length 13 m is moved from one corridor to the other and the height of the ceiling is 4 m. Can the rod be carried round the corner by **3**

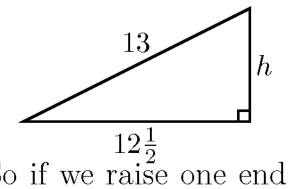
(i) pulling the rod on the floor?

Solution: As the shortest length that can touch A, B, and C is $12\frac{1}{2}$ m, any shorter length will fit through easily. A longer rod of 13 m will not be able to fit everywhere so will not get through.

(ii) lifting one end of the rod while the other end remains on the floor?

Explain your assertion.

Solution: If we raise one end as shown,



$$\begin{aligned} h^2 &= 13^2 - \left(12\tfrac{1}{2}\right)^2, \\ &= \frac{51}{4}, \\ h &\approx 3.57\,\text{m}. \end{aligned}$$

So if we raise one end by about 3.57 m, its length over the ground will be $12\frac{1}{2}$ m, and it will just fit through.

End of Section B

End of Paper