BAULKHAM HILLS HS PERSEVERE

YEAR 12 EXTENSION 1 MATHEMATICS ASSESSMENT JUNE 2009

TIME : 70 MINUTES

NAME		RESULT

DIRECTIONS

- Full working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Use black or blue pen only (*not pencils*) to write your solutions.
- No liquid paper is to be used. If a correction is to be made, one line is to be ruled through the incorrect answer.

QUESTION 1.

Differentiate

(a) $\tan^{-1} 4x$ 2

(b) $\sin^{-1}(e^{2x})$ 2

QUESTION 2.

Find the following :

(a) $\int \frac{dx}{9+25x^2}$ 2

(b) $\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{\sqrt{1-3x^2}}$ 3

QUESTION 3.

Prove that $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ where v is the velocity of a particle at time t. 2

QUESTION 4.

(a) Sketch $y = 2\sin^{-1}\frac{x}{3}$ stating the domain and range. 3

(b) Find the area between the curve and the y-axis in the first quadrant. 3

QUESTION 5.

Without the use of a calculator, evaluate the following showing all working:

a) $\cos^{-1}\left(-\frac{1}{2}\right)$ **2**

b) $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$ **2**

c) $\sin\left[\tan^{-1}\left(-\frac{2}{3}\right)+\cos^{-1}\left(\frac{12}{13}\right)\right]$ **3**

QUESTION 6.

Evaluate $\int_0^2 \sqrt{4-x^2}\,dx$ using the substitution $x = 2\sin\theta$. **4**

QUESTION 7.

Find $\int \frac{(\ln x)^2}{x}\,dx$ using the substitution $u = \ln x$. **3**

QUESTION 8.

A particle oscillates back and forth in a straight line with velocity v (ms^{-1}) in position x where $v^2 = 16x - 4x^2 + 20$.

a) Prove that the motion is simple harmonic motion. **3**
b) Find the extremities of the particle's motion. **2**
c) Find the period of the motion. **1**
d) Find the maximum speed reached by the particle. **2**

QUESTION 9.

A particle is moving along $x-axis$, starting from a position $2metres$ to the right of the origin with an initial velocity of $5ms^{-1}$ and an acceleration given by $a = 2x^3 + 2x$.

a) Show that $\dot{x} = x^2 + 1$ **2**
b) Hence find an expression for x in terms of time t. **3**

QUESTION 10.

A ball is projected from the top of a building 20 *metres* high with initial velocity 15 ms^{-1} and an angle of projection θ to the horizontal. The ball just clears a wall which is 6.25 *metres* high and 30 *metres* away from the foot of the building.
Let $g = 10ms^{-2}$.

a) Derive the equations for the horizontal and vertical displacement. **2**
b) Find the two possible angles of projection **3**
c) Find the exact time to reach the wall for the smaller angle of projection. **2**

Answers $\boxed{\text{Yr.12 3unit June 2009}}$ Total 51

1 a) $\frac{d}{dx}\tan^{-1}4x = \frac{4}{1+(4x)^2} = \frac{4}{1+16x^2}$ ✓ (2)

b) $\frac{d}{dx}\left(\sin^{-1}e^{2x}\right) = \frac{2e^{2x}}{\sqrt{1-(e^{2x})^2}} = \frac{2e^{2x}}{\sqrt{1-e^{4x}}}$ ✓ (2)

2 a) $\int \frac{dx}{9+25x^2} = \int \frac{dx}{25\left(\frac{9}{25}+x^2\right)} = \frac{1}{25}\cdot\frac{1}{\frac{3}{5}}\times\tan^{-1}\frac{x}{\frac{3}{5}}$ $\quad a=\frac{3}{5}$

$= \frac{5}{3}\times\frac{1}{25}\tan^{-1}\frac{5x}{3}+C = \frac{1}{15}\tan^{-1}\frac{5x}{3}+C$ ✓ (2)

b) $\int_0^{1/\sqrt3}\frac{dx}{\sqrt{1-3x^2}} = \int_0^{1/\sqrt3}\frac{dx}{\sqrt3\sqrt{\frac{1}{3}-x^2}} = \frac{1}{\sqrt3}\left[\sin^{-1}\sqrt3 x\right]_0^{1/\sqrt3}$

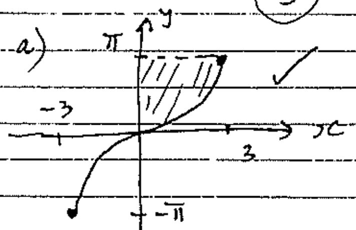
$= \frac{1}{\sqrt3}\left[\underbrace{\sin^{-1}1}_{\frac{\pi}{2}} - \sin^{-1}0\right] = \frac{\pi}{2\sqrt3}$ ✓ (3)

3 $\frac{d^2x}{dt^2} = \ddot{x} = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt}$ ✓ $\quad$ since $\frac{d}{dv}\left(\frac{1}{2}v^2\right)=v$

$= \frac{dv}{dx}\cdot\frac{d}{dv}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ (2)

4 $y = 2\sin^{-1}\left(\frac{x}{3}\right)$ $\quad -1 \le \frac{x}{3} \le 1$

a) $-3 \le x \le 3$



$D: -3 \le x \le 3$ ✓ (3)

$R: -\pi \le y \le \pi$ ✓

$x = 3\sin\frac{y}{2}$ $\quad y = 2\sin^{-1}\frac{x}{3}$

b) $A = \int^{\pi} 3\cdot\sin\frac{y}{2}\,dy = \left[-6\cos\frac{y}{2}\right]^{\pi}$ $\quad \frac{y}{2} = \sin^{-1}\frac{x}{3}$

4 cont.

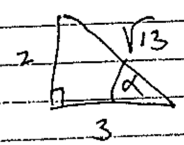
$\therefore A = -6\left[\underbrace{\cos\frac{\pi}{2}}_{0} - \underbrace{\cos 0}_{1}\right] = -6\times -1 = 6$ ✓ (3)

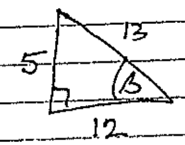
5 a) $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\frac{1}{2} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ ✓ (2)

b) $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\left(-\sqrt3\right) = -\underbrace{\tan^{-1}\sqrt3}_{\frac{\pi}{3}} = -\frac{\pi}{3}$ ✓ (2)

c) $\sin\left[\tan^{-1}\left(-\frac{2}{3}\right)+\cos^{-1}\left(\frac{12}{13}\right)\right] = \sin\left[\underbrace{-\tan^{-1}\left(\frac{2}{3}\right)}_{-\alpha}+\underbrace{\cos^{-1}\frac{12}{13}}_{\beta}\right]$

$= \sin(\beta-\alpha) = \sin\beta\cdot\cos\alpha - \cos\beta\cdot\sin\alpha$ ✓





$= \frac{5}{13}\cdot\frac{3}{\sqrt{13}} - \frac{12}{13}\cdot\frac{2}{\sqrt{13}}$ ✓ (3)

$= \frac{-9}{13\sqrt{13}}$ ✓

6 $\int_0^2 \sqrt{4-x^2}\,dx$ $\quad x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta\cdot d\theta$

~~$x=0$ $\theta=0$~~
~~$0 = 2\sin\theta$~~
$x = 2 = 2\sin\theta$
$1 = \sin\theta$
$\therefore x = \pi/2$ ✓ (1)

$= \int_0^{\pi/2}\sqrt{4-4\sin^2\theta}\,dx$ ✓

$= \int_0^{\pi/2} 2\sqrt{1-\sin^2\theta}\cdot 2\cos\theta\,d\theta$ ✓

$\cos 2\theta = 2\cos^2\theta - 1$

$= \int_0^{\pi/2} 4\cdot\cos^2\theta\,d\theta$ $\quad \frac{1}{2}(\cos 2\theta+1) = \cos^2\theta$ (4)

$= \int_0^{\pi/2} 4\times\frac{1}{2}(\cos 2\theta+1)\,d\theta = 2\left[\frac{1}{2}\sin 2\theta+\theta\right]_0^{\pi/2}$ ✓

$= \left[\sin 2\theta + 2\theta\right]_0^{\pi/2} = 0+\pi-0-0 = \pi$ ✓

(7) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$ $\qquad u = \ln x$ $\qquad \dfrac{du}{dx} = \dfrac{1}{x}$ $\therefore x\,du = dx$ ✓

$= \displaystyle\int \frac{u^2}{x}\cdot x\,du$ ✓

$= \displaystyle\int u^2\,du = \frac{u^3}{3} + c = \frac{(\ln x)^3}{3} + c$ ✓

(8) $v^2 = 16x - 4x^2 + 20$

a) $\dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = \ddot{x}$

since $v^2 = 16x - 4x^2 + 20$

$\therefore \tfrac{1}{2}v^2 = 8x - 2x^2 + 10$

$\therefore \ddot{x} = \dfrac{d}{dx}\left(8x - 2x^2 + 10\right) = 8 - 4x$ ✓ (3)

$\therefore \ddot{x} = -4(x-2)$ ✓ which is in the form

$\ddot{x} = -n^2(x-m)$ ✓ where $n = 2$

centre of the motion $x = m = 2$

$\therefore$ SHM

b) $v = 0 = 16x - 4x^2 + 20$ ✓ (2)

$0 = 4(4x - x^2 + 5)$

$0 = -(x^2 - 4x - 5) = -(x-5)(x+1)$

$x = 5 \quad x = -1$ extremities ✓

$x = -1 \qquad x = 2 \qquad x = 5$

c) $T = \dfrac{2\pi}{n} = \dfrac{2\pi}{2} = \pi$ ✓ (1)

(8) cont.

d) speed$_{MAX}$ is at $x = 2$ (centre) ✓ (2)

$\therefore v^2 = 16\times 2 - 4\times 2^2 + 20 = 36$

$\therefore |v| = \text{speed} = \sqrt{36} = \underline{\underline{6}}\ ms^{-1}$ ✓

(9) $t = 0 \quad x = +2\ m \quad v = 5\ ms^{-1}$

$\ddot{x} = a = 2x^3 + 2x$

a) $\ddot{x} = \dfrac{d}{dx}\left(\tfrac{1}{2}v^2\right) = 2x^3 + 2x$

$\tfrac{1}{2}v^2 = \displaystyle\int 2x^3 + 2x\,dx$

$\tfrac{1}{2}v^2 = \dfrac{x^4}{2} + x^2 + c$ (2)

$x = 2 \quad v = 5 \quad \therefore \tfrac{1}{2}\times 25 = \dfrac{2^4}{2} + 2^2 + c$

$\tfrac{1}{2} = c$ ✓

$\therefore \tfrac{1}{2}v^2 = \dfrac{x^4}{2} + x^2 + \tfrac{1}{2} \quad \therefore v^2 = x^4 + 2x^2 + 1$

$\therefore v^2 = (x^2+1)^2$

$\therefore v = \pm(x+1)$ ✓ but $t = 0 \quad v = +5$ and $v \neq 0$ $\quad v = x^2 + \ldots$ always +

justify + only

$\therefore v = +(x+1)$ only

$\therefore$ shown

b) $v = \dfrac{dx}{dt} = x^2 + 1$ $\qquad \therefore x = \dfrac{\tan t + 2}{1 - 2\tan t}$ ✓

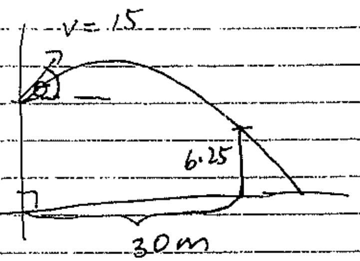
$\therefore \dfrac{dt}{dx} = \dfrac{1}{x^2+1} \quad \therefore t = \displaystyle\int \frac{1}{x^2+1}\,dx$ (3)

$t = \tan^{-1}x + c$ ✓

$t = 0 \quad x = 2$

$\therefore 0 = \tan^{-1}2 + c \quad \therefore c = -\tan^{-1}2 \quad \therefore t = \tan^{-1}x - \tan^{-1}$ ✓

(10)



$g = 10$

(a) $\ddot{x} = 0$

$\dot{x} = \int 0\,dt = c = v\cos\theta$

$x = \int v\cos\theta\,dt = vt\cos\theta + c$

$t=0 \quad x = 0 = c$

$\therefore \boxed{x = vt\cos\theta}$ ✓

$\ddot{y} = -g$

$\dot{y} = \int -g\,dt$

$\dot{y} = -gt + c$

$t=0 \quad \dot{y} = v\sin\theta$

$\therefore \dot{y} = -gt + v\sin\theta$

$y = \int -gt + v\sin\theta\,dt$

$y = -\frac{1}{2}gt^2 + vt\sin\theta + c$

$t=0 \quad y = 20 = c$ ✓

$\therefore \boxed{y = -\frac{1}{2}gt^2 + vt\sin\theta + 20}$

(2)

$\therefore$ (1) $\boxed{x = 15t\cos\theta}$

(2) $\boxed{y = -5t^2 + 15t\sin\theta + 20}$

(b) (1) $t = \frac{x}{15\cos\theta}$

(2) $y = -5\frac{x^2}{15^2\cos^2\theta} + 15\frac{x}{15\cos\theta}\cdot\sin\theta + 20$

$y = \frac{-5x^2}{15^2}(1+\tan^2\theta) + x\tan\theta + 20$

$y = \frac{-1}{45}x^2(1+\tan^2\theta) + x\tan\theta + 20$ ✓

$x = 30$

$y = 6.25$

$6.25 = \underbrace{\frac{-1}{45}\times 30^2}_{20}(1+\tan^2\theta) + 30\tan\theta + 20$

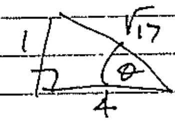
$6.25 = -\cancel{20} - 20\tan^2\theta + 30\tan\theta + \cancel{20}$

$20\tan^2\theta - 30\tan\theta + 6.25 = 0$ ✓

$\therefore \tan\theta = \frac{30 \pm \sqrt{30^2 - 4\times20\times6.25}}{2\times20} = \frac{30\pm20}{40} = \frac{5}{4} \text{ or } \frac{1}{4}$ (3)

$\therefore \boxed{\theta = 14^\circ 2' \text{ or } 51^\circ 20'}$ ✓

c) $t = \frac{x}{15\cos\theta}$



smaller $\tan\theta = \frac{1}{4}$

$\therefore \cos\theta = \frac{4}{\sqrt{17}}$ ✓

$t = \frac{x=30}{15\times\frac{4}{\sqrt{17}}} = \frac{30\sqrt{17}}{15\times4}$

$\therefore \boxed{t = \frac{\sqrt{17}}{2}}$ ✓ (2)

Total 51