YEAR 12 - EXTENSION 1
MATHEMATICS ASSESSMENT
JUNE 2010

TIME : 65 MINUTES

DIRECTIONS	▪ Full working should be shown in every question. Marks may be deducted for careless or badly arranged work. ▪ Use black or blue pen only (*not pencils*) to write your solutions. ▪ No liquid paper is to be used. If a correction is to be made, one line is to be ruled through the incorrect answer.	
QUESTION 1.	Differentiate a) $\tan^{-1}(e^x)$ b) $\cos^{-1}\sqrt{1-x^2}$	 2 2
QUESTION 2.	Without the use of a calculator, evaluate the following showing all working: a) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)-\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ b) $\sin^{-1}\sin\left(\frac{13\pi}{6}\right)$ c) $\sin\left(\sin^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)\right)$	 2 1 3
QUESTION 3.	Given $f(x)=3\sin^{-1}(x+1)$ a) State the Domain and Range of $f(x)$ b) Sketch $f(x)$ showing end points and intercepts.	 2 2
QUESTION 4.	An arrow is fired horizontally at $60ms^{-1}$ from the top of a $20m$ high wall. Taking $g=10\ ms^{-2}$ a) Show, using calculus, that the horizontal and vertical components of the arrows motion are given by $x=60t$, $y=-5t^2+20$ b) Find the time taken for the arrow to hit the ground. c) Find the distance that the point of impact will be from the base of the wall. d) To the nearest degree, find the acute angle with which the arrow will strike the ground.	 3 2 1 2

QUESTION 5.	The surface area of a sphere is increasing at a constant rate of $6cm^2\ per\ sec.$ Find the exact rate of increase of the radius at the instant when the radius is $5cm$	2
QUESTION 6.	Find the following a) $\int \frac{dx}{\sqrt{1-2x^2}}$ b) $\int \frac{dx}{x^2-4x+8}$	2 3
QUESTION 7.	Find $\int \sin^2 3x\, dx$	2
QUESTION 8.	A particle is moving on a straight line in such a way that its displacement x metres from the origin at time t seconds is given by $x = 5\sin 2t$ a) Show that the particle is moving in simple harmonic motion. b) Find the maximum speed of the particle. c) Find the acceleration of the particle when its displacement is $0.5m.$ d) Find the total distance travelled for the first 3 seconds Leave your answer to the nearest m.	1 1 1 2
QUESTION 9.	By using the substitution $u = e^{\frac{x}{2}}$, evaluate $\int_0^{\ln 3} \frac{e^{\frac{x}{2}}}{1+e^x}\,dx$	4
QUESTION 10.	An object moves so that its acceleration in terms of its displacement is given by $\frac{d^2x}{dt^2} = 10x - 2x^3$ a) Given $\frac{d^2x}{dx^2} = 10x - 2x^3$ and $v = 0$ when $x = 1$ Show that $v^2 = 10x^2 - x^4 - 9.$ b) Find the set of all possible values of x where motion can exist. c) Describe briefly what would happen if the motion had commenced from rest at $x = -1.$	2 3 2

~ END OF EXAM ~

Q1 a) $y = \tan^{-1} e^{x}$

$$\frac{dy}{dx} = \frac{e^{x}}{1+e^{2x}}$$ — ① — ①

b) $y = \cos^{-1}\sqrt{1-x^2}$

$$\frac{dy}{dx} = \frac{-\frac{1}{2}(1-x)^{-\frac{1}{2}} \times -2x}{\sqrt{1-(1-x^2)}}$$ — ①

$$= \frac{\left(\frac{x}{\sqrt{1-x^2}}\right)}{x}$$

$$= \frac{1}{\sqrt{1-x^2}}$$ — ①

Q2 a) $\cos^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

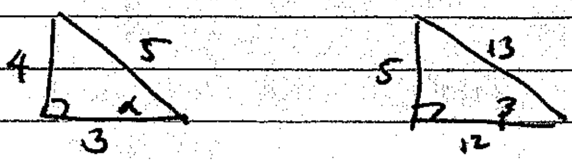
$= \cos^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\frac{1}{\sqrt{2}}$ — ①

$= \frac{\pi}{2}$ — ①

b) $\sin^{-1}\left(\sin\frac{13\pi}{6}\right) = \frac{\pi}{6}$ — ①

c) $\sin\left(\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13}\right)$

Let $\alpha = \sin^{-1}\frac{4}{5}$ $\quad \beta = \cos^{-1}\frac{12}{13}$



①

$\sin(\alpha+\beta)$

$= \sin\alpha\cos\beta + \sin\beta\cos\alpha$ — ①

$= \frac{4}{5} \times \frac{12}{13} + \frac{5}{13} \times \frac{3}{5}$

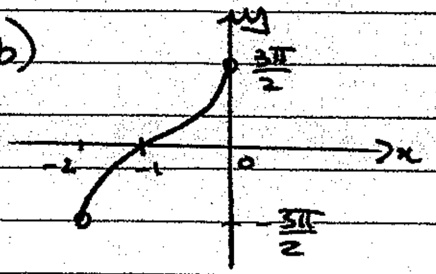
$= \frac{63}{65}$. — ①

Q3 a) $D: -1 \le x+1 \le 1$

$D: -2 \le x \le 0$ — ①

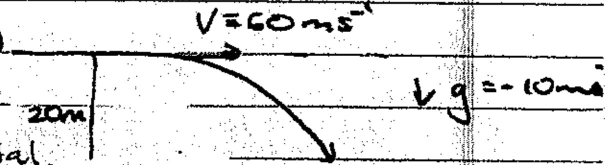
$R: -\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ — ①

b)



1mk shape
1mk all intercepts

Q4 a)



horizontal

$\ddot{x} = 0$

$\dot{x} = 60$

$x = 60t + C$

when $t = 0$; $x = 0$

$x = 60t$ — ①

vertical

$\ddot{y} = -10$

$\dot{y} = -10t + C$

when $t = 0$ $\dot{y} = 0$

$\dot{y} = -10t$

$y = -5t^2 + C$ — ①

when $t = 0$ $y = 20$

$20 = C$

$\therefore y = -5t^2 + 20$. — ①

b) hits ground when $y = 0$

$0 = -5t^2 + 20$

$t^2 = 4$ — ①

$t = \pm 2$ but $t > 0$

$\therefore t = 2$ — ①

c) $x = 60 \times 2$

$x = 120m$. — ①

Q4 d) at $t=2$

$\dot{x} = 60$

$\dot{y} = -10 \times 2 = -20$

$\therefore \tan\theta = \dfrac{20}{60}$ 1

$\theta = 18°26'$ (acute ∠) 1

Q5 $\dfrac{dA}{dt} = 6$

$\dfrac{dr}{dt} = \dfrac{dA}{dt} \times \dfrac{dr}{dA}$

$A = 4\pi r^2$

$\dfrac{dA}{dr} = 8\pi r$ 1

when $r = 5$

$\dfrac{dr}{dt} = 6 \times \dfrac{1}{8\pi \times 5}$

$\dfrac{dr}{dt} = \dfrac{3}{20\pi}$ 1

Q6 a) $\displaystyle\int \frac{dx}{\sqrt{1-2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{1}{2} - x^2}}$

$= \dfrac{1}{\sqrt{2}} \sin^{-1}(\sqrt{2}\,x) + C$

1 1

b) $\displaystyle\int \frac{dx}{x^2 - 4x + 4 + 4} = \int \frac{dx}{4 + (x-2)^2}$

$= \dfrac{1}{2} \tan^{-1}\left(\dfrac{x-2}{2}\right) + C$

1 1 1

Q7 $\cos 2A = 1 - 2\sin^2 A$

$\sin^2 A = \dfrac{1 - \cos 2A}{2}$

1

$\displaystyle\int \sin^2 3x\, dx = \frac{1}{2} \int 1 - \cos 6x\, dx$

$= \dfrac{1}{2}\left[x - \dfrac{1}{6}\sin 6x\right] + C$ 1

$= \dfrac{x}{2} - \dfrac{1}{12}\sin 6x + C.$

Q8 a) $x = 5\sin 2t$

$\dot{x} = 2 \times 5 \cos 2t$

$\ddot{x} = -4 \times 5 \sin 2t$

$\ddot{x} = -4x$

$\therefore$ SHM 1

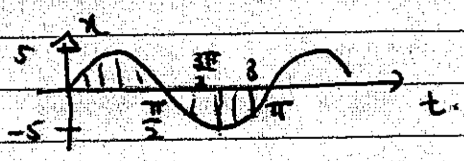
b) Since $\dot{x} = 10\cos 2t$

max speed $= 10\,ms^{-1}$ 1

c) $\ddot{x} = -4 \times 0.5$

$\ddot{x} = -2\,ms^{-2}$ 1

d)



upto $\frac{3\pi}{2}$ secs, it has travelled 15m

the remaining trip $= 5\sin(3) - (-5)$

$= 5.70\ldots$

$\therefore$ total distance $= 20.70$ m 1

$= 21$m 1

Q9 $u = e^{\frac{x}{2}}$

$\frac{du}{dx} = \frac{1}{2} e^{\frac{x}{2}} dx$

if $x = \ln 3$ $\quad u = e^{\frac{\ln 3}{2}} = \sqrt{3}$ (1)

$x = 0 \quad u = e^0 = 1$

$2\int_0^{\ln 3} \frac{\frac{1}{2} e^{\frac{x}{2}}}{(1+e^x)} dx$ —— (1)

$= 2\int_1^{\sqrt{3}} \frac{du}{1+u^2}$

$= 2\left[\tan^{-1} u\right]_1^{\sqrt{3}}$ —— (1)

$= 2\left[\tan^{-1}\sqrt{3} - \tan^{-1} 1\right]$

$= 2\left[\frac{\pi}{3} - \frac{\pi}{4}\right]$

$= \frac{\pi}{6}$ —— (1)

Q10 $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 10x - 2x^3$ —— (1)

$\frac{1}{2}v^2 = 5x^2 - \frac{1}{2}x^4 + C$

when $v = 0$; $x = 1$

$0 = 5 - \frac{1}{2} + C$

$C = -4\frac{1}{2}$ —— (1)

$\frac{1}{2}v^2 = 5x^2 - \frac{1}{2}x^4 - 4\frac{1}{2}$

$v^2 = 10x^2 - x^4 - 9$

b) Since $v = \pm\sqrt{10x^2 - x^4 - 9}$

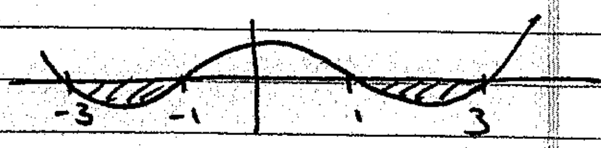
then $10x^2 - x^4 - 9 \geq 0$ —— (1)

$x^4 - 10x^2 + 9 \leq 0$

$(x^2-9)(x^2-1) \leq 0$

$(x-3)(x+3)(x-1)(x+1) \leq 0$

values of x —— (1)



$\therefore$ motion can only exist at

$-3 \leq x \leq 1$ or

$1 \leq x \leq 3$ —— (1)

correct region

c) when $x = -1$; $v = 0$

$a = 10(-1) - 2(-1)^3$

$a = -8$

$\therefore$ at $x = -1$ it will start to accelerate to the left.

at $x = -3$; $v = 0$

$a = 10(-3) - 2(-3)^3$

$a = 24$

$\therefore$ at $x = -1$ it will move faster to the left, then slows down and will stop at $x = -3$, and then move faster to the right, then slows down and stops at $x = -1$. (2)

It will repeat But since $\ddot{x} \neq -n^2 x$ then it is NOT in simple harmonic motion

If SHM (-1).