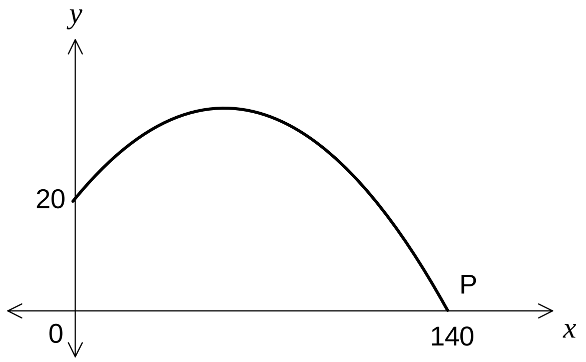
YEAR 12
EXTENSION 1 MATHEMATICS
ASSESSMENT TASK
JUNE 2012

TIME : 60 MINUTES (PLUS 5 MINUTES READING TIME)

QUESTION 1.	Let $f(x) = 2x^3 + 2x - 1$ a) Show that $f(x)$ has a root between $x = 0$ and $x = 1$ b) By considering $f\,'(x)$, explain why there is exactly one root for $f(x)$ c) Taking $x = 0.5$ as an initial approximation, use one application of Newton's Method to find a closer approximation	 **2** **1** **2**
QUESTION 2.	Evaluate $\displaystyle\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx$ by using the substitution $u = x^2 + 1$	**3**
QUESTION 3.	Find $\displaystyle\int \frac{sin2x}{3sin^2x - cos^2x}\,dx$ by using the substitution $u = \sin^2 x$	**3**
QUESTION 4.	The acceleration of a particle moving along the x −axis is given by $\ddot{x} = 4x + 2$, where x is its displacement (in metres) from O. Initially, the particle is at O and its velocity is 1 m/s. a) Show that its velocity is given by $v = 2x + 1$ b) Find when the particle reaches a velocity of 9 m/s	 **3** **3**
QUESTION 5.	A particle moves such that its displacement, x metres , from the origin is given by $x = \cos 3t - \sin 3t$ where *t* is the time in seconds a) By differentiation, show that the particle is moving in simple harmonic motion b) Express x in the form $Rcos(3t + \alpha)$ c) State the amplitude and period of the motion d) Find the maximum speed of the particle.	 **2** **2** **2** **2**

QUESTION 6.	a) Show that $\frac{d}{dx}\left[tan^{-1}\left(\frac{x}{2}\right)+\frac{2x}{x^2+4}\right]=\frac{16}{\left(x^2+4\right)^2}$ b) Hence or otherwise, evaluate $\int_{-2}^{2}\frac{dx}{(x^2+4)^2}$, in its exact form	**3** **2**
QUESTION 7.	A spherical balloon is being inflated so that its surface area is increasing at a constant rate of 15 mm^2/s. When the radius is 5 mm, find: a) The rate at which the radius is increasing b) The rate at which the volume is increasing	 **2** **2**
QUESTION 8.	A stone is projected from a point at the top of a vertical cliff, 20 metres above sea level. The angle of projection is θ° above horizontal and the initial speed is 35 m/s. The stone hits the sea at a point P, 140 metres from the base of the cliff. The equations for the horizontal components of the motion are given as: $\ddot{x}=0$ $\quad\quad$ $\dot{x}=35cos\theta$ $\quad\quad$ $x=35tcos\theta$ (Do not prove these results)  a) Given that $\ddot{y}=-g$, derive the equations for $\dot{y}$ and y b) Find the two possible values for θ, correct to the nearest degree. (Use g= 10 m/s^2) c) Another stone is projected horizontally with initial speed V m/s from the same point at the top of the cliff and falls into the sea at the same point P. Find the speed of this stone just before it enters the sea.	**2** **4** **3**

<u>End of exam</u>

Year 12
XI June 2012.

SOLUTIONS

Note: TOTAL/43.

Q1. a) $f(x) = 2x^3 + 2x - 1$

$f(0) = 0 + 0 - 1 < 0$

$f(1) = 2 + 2 - 1 > 0$ (1)

Since $f(x)$ is continuous, and $f(0), f(1)$ have opposite signs, there is a root between $x=0, x=1$

MUST HAVE 'CONTINUOUS', ... (1)

b) $f'(x) = 6x^2 + 2 > 0$ for all values of x.

$\therefore f(x)$ is always increasing and can only cross the x-axis once. (1)

c) $f(0.5) = 2(0.125) + 2(0.5) - 1 = 0.25$

$f'(0.5) = 6(0.25) + 2 = 3.5$ (1)

$\therefore \quad a = 0.5 - \dfrac{0.25}{3.5}$

$= 0.429$ (3dp) or $3/7$ (exact) (1)

Q2. $\displaystyle\int_0^{\sqrt{3}} x . \sqrt{x^2+1}. \, dx$

$u = x^2 + 1$

$\frac{1}{2} . du = \cancel{2}x . dx$

$x = 0 : u = 1$

$x = \sqrt{3} : u = 4$

$= \displaystyle\int_1^4 \sqrt{u} . \frac{1}{2} . du$ ← (1)

$= \frac{1}{2}\left[\frac{2}{3} u^{3/2}\right]_1^4$

$= \frac{1}{3}\left[u^{3/2}\right]_1^4$ ← (1)

$= \frac{1}{3}(8 - 1)$

$= \dfrac{7}{3}$ ← (1)

Q3. $\displaystyle\int \frac{\sin 2x}{3\sin^2 x - \cos^2 x} \cdot dx$

$$u = \sin^2 x$$
$$du = 2\sin x \cos x \cdot dx$$
$$= \sin 2x \cdot dx \quad \leftarrow (1)$$

$= \displaystyle\int \frac{\sin 2x}{4\sin^2 x - 1} \cdot dx \quad \leftarrow (1)$

$= \displaystyle\int \frac{du}{4u - 1}$

$= \frac{1}{4} \ln (4u - 1) + c$

$= \frac{1}{4} \ln (4\sin^2 x - 1) + C \quad \leftarrow (1)$

Q4. a) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x + 2$

$\frac{1}{2} v^2 = \displaystyle\int 4x + 2 \, . \, dx$

$v^2 = 2\displaystyle\int 4x + 2 \, . \, dx$

$v^2 = 2(2x^2 + 2x) + c \quad \leftarrow (1)$

Sub. $x = 0$, $v = 1$

$1 = 0 + c \quad \therefore c = 1$

$v^2 = 4x^2 + 4x + 1 = (2x+1)^2 \quad \leftarrow (1)$

BUT initially $x = 0$, $v = 1$ so it moves to the right (where $x > 0$). Then $\ddot{x} > 0$ and v can only increase, so will always be positive.

$\therefore \quad v = 2x + 1. \quad \leftarrow (1)$

OR because $x = 0, v = 1$ only satisfies $v = 2x + 1$ and doesn't satisfy $v = -(2x+1)$

b) When $v = 9$, $2x+1 = 9$, $x = 4$ $\leftarrow$ (1)

$$v = \frac{dx}{dt} = 2x+1$$

$$\frac{dt}{dx} = \frac{1}{2x+1}$$

Time taken, $t = \displaystyle\int_0^4 \frac{1}{2x+1} \cdot dx$ $\leftarrow$ (1)

$$= \frac{1}{2}\Big[\ln(2x+1)\Big]_0^4$$

$$= \frac{1}{2}(\ln 9 - \ln 1) \quad \leftarrow (1)$$

$$= \underline{\frac{1}{2}\ln 9 \quad \text{or} \quad \ln 3}$$

Q5. a) $x = \cos 3t - \sin 3t$

$\dot{x} = -3\sin 3t - 3\cos 3t$ $\leftarrow$ (1)

$\ddot{x} = -9\cos 3t + 9\sin 3t$

$= -9(\cos 3t - \sin 3t)$

$= -9x$ $\leftarrow$ (1)

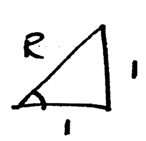
$= -n^2 x$ (with $n = 3$) $\therefore$ SHM.

b) $x = \cos 3t - \sin 3t = R\cos(3t+\alpha)$

$= R\cos 3t\cos\alpha - R\sin 3t\sin\alpha$

$\cos\alpha = 1/R$

$\sin\alpha = 1/R$



$R = \sqrt{2}$ (1)

$\alpha = \pi/4$ (1)

$\therefore \quad x = \sqrt{2}\cos(3t + \pi/4)$

c) Amplitude $= \sqrt{2}$ —— (1)

Period $= 2\pi/3$ —— (1)

d) $v = \dfrac{dx}{dt} = -3\sqrt{2}\sin(3t+\pi/4)$ (1) $\Rightarrow$ Max $v = \underline{3\sqrt{2} \text{ m/s}}$ (1)

Q6. a) $\frac{d}{dx}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right)$

$= \frac{2}{4+x^2} + \frac{(x^2+4).2 - 2x.2x}{(x^2+4)^2}$

(1)

$= \frac{2}{4+x^2} + \frac{8-2x^2}{(x^2+4)^2}$ ← (1)

$= \frac{2x^2+8 + 8-2x^2}{(x^2+4)^2}$ ← (1)

$= \frac{16}{(x^2+4)^2}$

b) $\int_{-2}^{2} \frac{dx}{(x^2+4)^2} = \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{2x}{x^2+4}\right)_{-2}^{2}$ ← (1)

$= \frac{1}{16}\left(\left(\tan^{-1}1 + \frac{1}{2}\right) - \left(\tan^{-1}(-1) - \frac{1}{2}\right)\right)$

$= \frac{1}{16}\left(\frac{\pi}{4} + \frac{1}{2} + \frac{\pi}{4} + \frac{1}{2}\right)$

$= \frac{1}{16}\left(\frac{\pi}{2} + 1\right)$ ← (1)

Q7. $V = \frac{4}{3}\pi r^3$ $\qquad \frac{dV}{dr} = 4\pi r^2$ $\qquad \frac{ds}{dt} = 15$ (constant)

$S = 4\pi r^2$ $\qquad \frac{dS}{dr} = 8\pi r$

a) $\frac{dr}{dt} = \frac{dS}{dt} \cdot \frac{dr}{dS}$

$= 15 \cdot \frac{1}{8\pi r}$ $\qquad \leftarrow (1)$

$= \frac{15}{8\pi \times 5}$

$= \frac{3}{8\pi}$ mm/s. $\qquad \leftarrow (1)$

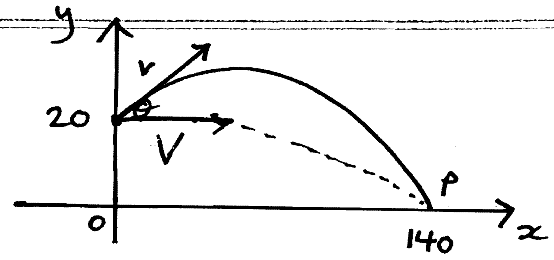
b) $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$= 4\pi r^2 \cdot \frac{3}{8\pi}$ $\qquad \leftarrow (1)$

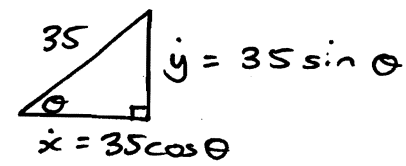
$= 100\pi \cdot \frac{3}{8\pi}$ $\qquad$ (when $r = 5$)

$= \frac{300}{8}$ mm^3/s $\qquad \leftarrow (1)$

Q8.



Initial vel :



$t = 0 \ , \ x = 0 \ , \ y = 20$

a) $\ddot{y} = -g$

$\dot{y} = \int -g \, . \, dt$

$= -gt + c$

$\left.\begin{array}{l} t=0 \\ \dot{y} = 35 \sin\theta \end{array}\right] \quad 35 \sin\theta = c$

$\underline{\dot{y} = -gt + 35 \sin\theta}$

$y = \int -gt + 35 \sin\theta \, . \, dt$

$= -\frac{1}{2}gt^2 + 35 t \sin\theta + c$

$\left.\begin{array}{l} t=0 \\ y=20 \end{array}\right] \quad 20 = 0 + 0 + c$

$\therefore \ \underline{y = -\frac{1}{2}gt^2 + 35 t \sin\theta + 20}$

b) When $y = 0$, $x = 35$

$35 t \cos\theta = 140$

$t = \frac{4}{\cos\theta} = 4 \sec\theta \quad$ — (1)

$y = -5t^2 + 35t \sin\theta + 20$

$0 = -5(16 \sec^2\theta) + 35(4\sec\theta) . \sin\theta + 20$

$0 = -80 \sec^2\theta + 140 \tan\theta + 20$

$(\div -20) \quad 4 \sec^2\theta - 7 \tan\theta - 1 = 0$

$\}\leftarrow$ (1) Any equivalent

$4(1 + \tan^2\theta) - 7\tan\theta - 1 = 0$

$4 \tan^2\theta - 7 \tan\theta + 3 = 0$

$(4 \tan\theta - 3)(\tan\theta - 1) = 0 \quad \leftarrow$ (1)

$\tan\theta = \frac{3}{4} \quad$ or $\quad \tan\theta = 1$

$\therefore \ \underline{\theta \doteq 37^\circ \quad \text{or} \quad \theta = 45^\circ}$

c) $\dot{x} = V$

$Vt = 140$

$t = 140/V$

$\ddot{y} = -g$

$\dot{y} = -gt = -10t$

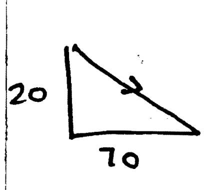
$y = -\frac{1}{2}gt^2 + 20 = -5t^2 + 20$

When $y=0$, $5t^2 = 20$, $\underline{t=2}$ $(t>0)$

(1)

Then $\dot{y} = -10(2) = -20$

$\dot{x} = V = 70$



Velocity at impact $= \sqrt{20^2+70^2}$

$= \underline{\sqrt{5300}}$ m/s (1)