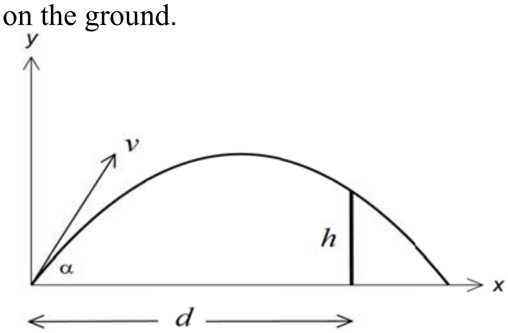
Question No	Question	Mark
1	a) Differentiate $(x^2 + 2x + 2)e^{-x}$	3
	b) Hence evaluate $\int_1^2 x^2 e^{-x} dx$	2
2	Find $\int \frac{\cos x}{1 + \sin^2 x} dx$ by using the substitution $u = \sin x$	3
3	a) Let $P(\theta) = 2\sin\theta - \theta$. Show that $P(\theta)$ has a root between 1 and 2.	2
	b) Taking $\theta = 1.8$ as an initial approximation for the solution to the equation $2\sin\theta = \theta$ between $\frac{\pi}{2}$ and π, use one application of Newton's Method to find a closer approximation.	2
4	A spherical balloon is being inflated so that its surface area is increasing at a steady rate of $12\text{cm}^2/\text{s}$.	
	a) Show that $\frac{dr}{dt} = \frac{3}{2\pi r}$ where r is the radius of the balloon.	2
	b) Find the rate of increase of the volume at the instant that the surface area is $1600\,\pi\text{ cm}^2$.	3
5	(i) Prove that $\frac{d^2x}{dt^2} = \frac{d(\frac{1}{2}v^2)}{dx}$ where v and x are the velocity and displacement of a particle respectively.	2
	(ii) A particle moves in a straight line with an acceleration given by $\ddot{x} = -\frac{1}{2}e^{-x}$. If $v = 1$ and $x = 0$ when $t = 0$, find the	
	a) velocity in terms of x	2
	b) displacement in terms of t	3

6	The speed v m/s of a particle moving along the x axis is given by $v^2 = 18 + 32x - 8x^2$, where x is the distance from the origin. a) Prove that the particle is in Simple Harmonic Motion. b) Find its centre of motion and its period. c) What is the amplitude?	 2 2 2
7	When the trigger of an aerosol can is depressed, the rate of decrease of the gas pressure in the can is given by $\frac{dp}{dt} = k\sqrt{p-1}$ where p is the pressure (in atmospheres) in the can after t seconds. Initially the pressure is 10 atmospheres and after 10 seconds the pressure is 5 atmospheres. a) Calculate the value of k. b) Find the pressure in the can after using it for 20 seconds.	 3 2
8	A ball is projected at an angle of elevation α with speed V from a point on the ground.  i) The equations of motion are given as: $x = Vt\cos\alpha$ $y = Vt\sin\alpha - \frac{1}{2}gt^2.$ Prove that the maximum height reached is $\frac{V^2\sin^2\alpha}{2g}$ ii) The ball just clears a wall of height h m at a distance of d m from the initial point of projection. Prove that the maximum height reached by the ball is: $\frac{1}{4}\left(\frac{d^2\tan^2\alpha}{d\tan\alpha - h}\right)$	 2 3

End of Task

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \; if \; n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \; x > 0$

BAULKHAM HILLS HIGH SCHOOL

2013

HSC Assessment Task 3

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Attempt all questions

Total marks – 40

This paper consists of EIGHT questions on pages 3-4.

Attempt all questions

Answer each question on the appropriate page in the answer booklet

A page of standard integrals is on page 2.

Student's Number:.......................... **Teacher:**....TEACHER....

BAULKHAM HILLS HS PERSEVERE

Year 12
HSC Assessment Task - 3
June 2013
MATHEMATICS EXTENSION 1

Marking Cover Sheet

Basic Skills		Problem Solving	
Question	Mark	Question	Mark
1 a)	/3	1 b)	/2
		2	/3
3 b)	/2	3 a)	/2
4 a)	/2	4 b)	/3
5 i)	/2	5 ii) a, b	/5
6 a)	/2	6 b), c)	/4
7 b)	/2	7a)	/3
8 i)	/2	8 ii)	/3
SUB TOTAL	/15	SUB TOTAL	/25
TOTAL	/40		

190

Question 1 **BOS#:**__________

a) $\frac{d}{dx}(x^2+2x+2)e^{-x}$

$= e^{-x}[2x+2] + (x^2+2x+2)(-1)e^{-x}$

$= e^{-x}(\cancel{2x}+\cancel{2}-x^2-\cancel{2x}-\cancel{2})$ 3

$= -x^2e^{-x}$

b) $\int_1^2 e^{-x}.x^2 = \left[-(x^2+2x+2)e^{-x}\right]_1^2$

$= -10e^{-2}+5e^{-1}$

$= \frac{5}{e} - \frac{10}{e^2}$ 2

$= 1.839 - 1.3534$

$= 0.4856.$

You may ask for extra writing paper if you need more space to answer question 1

Question 2 **BOS#:**____________________

$u = \sin x$

$\frac{du}{dx} = \cos x$ ✓

or

$\int \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{du}{1+u^2}$ ✓

3

$= \tan^{-1}\left(\frac{u}{1}\right) + C$ ✓

$= \tan^{-1}(\sin x) + C$ ✓ ← Replace u and C included.

You may ask for extra writing paper if you need more space to answer question 2

Question 3 **BOS#:**____________________

a) $P(\theta) = 2\sin\theta - \theta$

$P(1) = 2\sin(1) - 1 = 0.6829$

$P(2) = 2\sin(2) - 2 = -0.1814$

$P(1)$ and $P(2)$ have opposite signs and $P(\theta)$ is continuous in $1 \le \theta \le 2$, $\exists$ a root between 1 and 2.

b) $\theta_0 = 1.8$

$\theta_1 = \theta_0 - \frac{P(\theta)}{P'(\theta)}$

$= 1.8 - \frac{[2\sin(1.8) - 1.8]}{[2\cos(1.8) - 1]}$ ✓

or

$= 1.8 - \frac{0.1477}{-1.4544}$

$= 1.9$ ~~(3 d.p.)~~ ✓

You may ask for extra writing paper if you need more space to answer question 3

Question 4 **BOS#:**

a) $A = 4\pi r^2$ $\quad V = \frac{4}{3}\pi r^3$

$\frac{dA}{dt} = 12 \text{ cm}^2/s$

$\frac{dA}{dt} = 4\pi\, 2r \cdot \frac{dr}{dt}$ ✓ $\Rightarrow$ $\cancel{12}^{3} = \cancel{4}\pi \cdot 2r \cdot \frac{dr}{dt}$ ✓ (2)

$\frac{dr}{dt} = \frac{3}{2\pi r}$

b) $\frac{dv}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3}r^2 \cdot \frac{dr}{dt}$

$= 4\pi r^2 \cdot \frac{dr}{dt}$ ✓

when $A = 1600\pi \text{ cm}^2$ $\quad \cancel{1600}^{400}\cancel{\pi} = \cancel{4}\cancel{\pi} r^2$

$r = 20$ ✓

$\therefore \frac{dv}{dt} = \cancel{1600}^{4}\cancel{\pi} \cdot \frac{3}{\cancel{2\pi} \times \cancel{20}} = 120.$ (3)

$\therefore$ rate of increase $= 120 \text{ cm}^3/s$ ✓

Question 5 **BOS#:**

(i) $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right).$

LHS $= \frac{dv}{dt}$ ✓

$= \frac{dv}{dx} \cdot \frac{dx}{dt}.$

$= v \cdot \frac{dv}{dx}$ ✓ $\quad$ where $\frac{dx}{dt} = v$

$= \frac{d\left(\frac{1}{2}v^2\right)}{dx}$

$=$ RHS.

(ii) a) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{2}e^{-x}$ ✓

$\frac{1}{2}v^2 = -\frac{1}{2}\int e^{-x}\,dx \Rightarrow \frac{1}{2}v^2 = +\frac{1}{2}e^{-x} + k.$ (1)

when $x = 0$, $v = 1 \Rightarrow k = 0.$

$\therefore v^2 = e^{-x}$ —(1).

$\therefore v = e^{-\frac{1}{2}x}$ ✓

b) from (1) $\frac{dx}{dt} = e^{-\frac{1}{2}x}$ ✓ $\Rightarrow t = \int e^{\frac{1}{2}x}\,dx$

$t = 2e^{\frac{1}{2}x} + c$

$t = 0, x = 0 \Rightarrow c = -2$ ✓ (1)

$\therefore 2e^{\frac{1}{2}x} = t + 2 \Rightarrow e^{\frac{1}{2}x} = \frac{t}{2} + 1$

$\frac{1}{2}x = \ln\left(\frac{t}{2} + 1\right)$

$x = 2\ln\left(\frac{t}{2} + 1\right)$ ✓ (1)

Question 6 **BOS#:** ______________________

a) $v^2 = 18 + 32x - 8x^2$

$\frac{1}{2}v^2 = -4x^2 + 16x + 9$

differentiate w.r.t x

$\frac{d(\frac{1}{2}v^2)}{dx} = -8x + 16$ ✓

$\ddot{x} = -8(x-2)$ ✓ $\because \frac{d^2x}{dt^2} = \frac{d(\frac{1}{2}v^2)}{dx}$.

$\ddot{X} = -n^2 X$ when $X = x - 2$, and $n = 2\sqrt{2}$

$\therefore$ The particle is in SHM.

b) Centre of motion is $X = 0$, ie $x = 2$ ✓

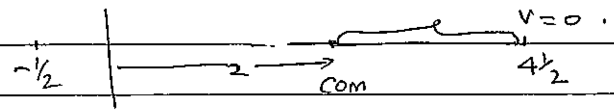
period $= \frac{2\pi}{n} = \frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$ s ✓

c) Amplitude is the max displacement from the centre of motion. At this point $v = 0$.

$4x^2 - 16x - 9 = 0$ ✓

$4x^2 - 18x + 2x - 9 = 0$

$2x(2x-9) + 1(2x-9) = 0 \Rightarrow x = 4\frac{1}{2}; -\frac{1}{2}$.



amplitude $= 4\frac{1}{2} - 2$

$= 2\frac{1}{2}$ cm ✓

You may ask for extra writing paper if you need more space to answer question 6

Question 7 **BOS#:** ______________________

a) $\frac{dP}{dt} = k\sqrt{P-1}$ when $t = 0$ $P = 10$; $t = 10$ $P = 5$

$\int \frac{dP}{\sqrt{P-1}} = k \int dt$

$kt = \int (P-1)^{-\frac{1}{2}} dP$ ✓

$kt = 2(P-1)^{\frac{1}{2}} + C$

$0 = 2 \times 3 + C \Rightarrow C = -6$ ✓

$kt = 2(P-1)^{\frac{1}{2}} - 6$

when $t = 10$ $P = 5$ $10k = 2 \times 2 - 6$

$k = -\frac{1}{5}$ ✓

b) $-\frac{1}{5}t = 2\sqrt{P-1} - 6$.

when $t = 20$

$-\frac{1}{5} \times 20 = 2\sqrt{P-1} - 6$ ✓

$\cancel{2} = \cancel{2}\sqrt{P-1}$

$P - 1 = 1$

$P = 2$ ✓

$\therefore$ after 20 s pressure in the can is 2 Atmospheres.

You may ask for extra writing paper if you need more space to answer question 7

Question 8 **BOS#:** ______________________

i) $x = Vt\cos\alpha$ —① $y = Vt\sin\alpha - \frac{1}{2}gt^2$ —②

$\dot{x} = V\cos\alpha$ $\dot{y} = V\sin\alpha - gt$.

at the highest point $\dot{y} = 0$. ie

$\therefore$ $V\sin\alpha - gt = 0$.

$t_{max} = \frac{V\sin\alpha}{g}$. ✓

In this time height reached is y.

$\therefore$ $y_{max} = V.\sin\alpha\left(\frac{V\sin\alpha}{g}\right) - \frac{1}{2}g\left(\frac{V^2\sin^2\alpha}{g^2}\right)$ ✓

$= \frac{V^2\sin^2\alpha}{g} - \frac{V^2}{2}\frac{\sin^2\alpha}{g}$

$= \frac{V^2\sin^2\alpha}{2g}$. — A

(ii) from ① $t = \frac{x}{V\cos\alpha}$; sub this in ②

$y = V\sin\alpha . \frac{x}{V\cos\alpha} - \frac{1}{2}g\frac{x^2}{V^2\cos^2\alpha}$

$y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{V^2\cos^2\alpha}$ — ③ ✓

when the ball ~~pass~~ clears the wall $x = d, y = h$

sub in ③

$h = d\tan\alpha - \frac{1}{2}g\frac{d^2}{V^2\cos^2\alpha}$.

$\Longrightarrow$ $\frac{V^2}{g} = \frac{d^2}{(d\tan\alpha - h)2\cos^2\alpha}$ ✓

Sub $\frac{v^2}{g}$ in (A')

$$y_{max} = \frac{\sin^2\alpha}{2} \cdot \frac{d^2}{(d\tan\alpha - h)2\cdot\cos^2\alpha} \checkmark$$

$$= \frac{d^2\tan^2\alpha}{4(d\tan\alpha - h)}$$

$$= \frac{1}{4}\left[\frac{d^2\tan^2\alpha}{d\tan\alpha - h}\right]$$

You may ask for extra writing paper if you need more space to answer question 8