BAULKHAM HILLS HIGH SCHOOL

**2014**

YEAR 12

TERM 2 ASSESSMENT TASK

# Mathematics Extension 1

**General Instructions**

- Reading time – 3 minutes
- Working time – 65 minutes
- Write using black or blue pen
- Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All relevant mathematical reasoning and/or calculations must be shown

**Total marks – 36**

This exam consists of 9 questions on pages 2-4.

Attempt all questions in the booklet provided.

**36 marks**
**Attempt all questions**

Answer each question in the answer booklet. Each answer sheet must show your BOS#. Extra Paper is available.

Your responses should include relevant mathematical reasoning and/or calculations.

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**Question 1**

Sylvia knows that the equation $y = x^4 - 2x^2 + x - 3$ has a root between $x$=1 and $x$=2. She was asked to use the halving the interval method to estimate this root to the nearest integer. She correctly calculates $f(1) = -3$ and $f(2) = 7$. Sylvia then concluded that the root is closer to $x = 1$ than $x = 2$. Is Sylvia correct? Justify your answer. **2**

**Question 2**

Find $\displaystyle\int \frac{dx}{x + \sqrt{x}}$ using the substitution $u = \sqrt{x}$. **3**

**Question 3**

Evaluate the following definite integral using the substitution $u = \sin x$ **3**

$$\int_0^{\frac{\pi}{2}} \cos x\sqrt{\sin^5 x}\, dx$$

**Question 4**

It is known that $\log_e x + \sin x = 0$ has a root close to $x$=0.5. Use one application of Newton's method (taking $x$=0.5 as a first approximation) to find a better approximation to the root correct to two decimal places. **2**

**CONTINUED ON THE NEXT PAGE**

**Question 5**

(a) Sketch the graph of the function $y = \log_e(x - 2)$ **1**

(b) The region bounded by $y = \log_e(x - 2)$, $x = 0$, $y = 0$ and $y=h$ is rotated about the $y$ axis to create a bowl. **3**

Find the exact volume of the bowl in terms of $h$.

(c) The bowl is placed with its axis vertical and water is poured into the bowl at the rate of 50 $cm^3$ per second. **2**

Find the rate at which the water level is rising when the depth of water is 1.5$cm$, giving your answer correct to 3 decimal places.

**Question 6**

A particle is moving on a straight line. At time $t$ seconds it has displacement $x$ metres from a fixed point $O$ on the line, velocity $v\,ms^{-1}$ and acceleration $a\ ms^{-2}$. The particle starts from $O$ and at time $t$ seconds, $v = (1 - x)^2$.

(a) Find an expression for $a$ in terms of $x$. **2**

(b) Find an expression for $x$ in terms of $t$. **2**

(c) Find the time taken for the particle to slow down to a speed of 1% of its initial speed. **2**

**CONTINUED ON THE NEXT PAGE**

**Question 7**

A cat sitting at the top of a wall 3.2$m$ high sees a mouse on the ground 4$m$ from the bottom of the wall. The cat jumps horizontally from the top of the wall with an initial velocity of $V\ ms^{-1}$. Taking acceleration due to gravity to be 10 $ms^{-2}$ and ignoring air resistance, the horizontal displacement equation of motion is:

$x = Vt$. DO NOT PROVE THIS

(a) Show that the vertical displacement equation of motion is $y = 3.2 - 5t^2$ **1**

(b) Find the time taken for the cat to reach the ground. **1**

(c) If the cat wants to land on the mouse calculate the speed at which the cat must jump horizontally from the wall. **1**

**Question 8**

A particle moving in simple harmonic motion has its acceleration given by $\ddot{x} = -9x$ where $x$ is the displacement in metres of the particle from the origin after $t$ seconds.

(a) Show that $x = -a\sin(3t + \alpha)$ is a possible equation of motion where $a > 0$ and $\alpha$ are constants and $\alpha$ is acute. **2**

(b) Initially, the particle's acceleration is $18ms^{-2}$ and after $\frac{5\pi}{18}$ seconds the velocity is $12ms^{-1}$. Show that the amplitude is 4 metres. **2**

(c) Find the greatest speed and where it reaches this speed. **2**

(d) How many times does the particle change direction in the first 4 seconds? **2**

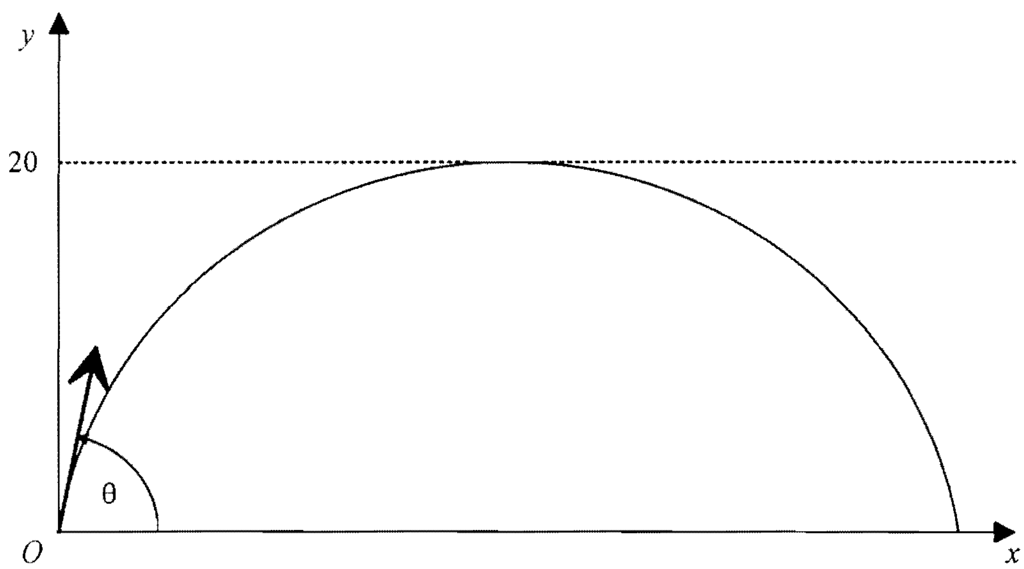
**CONTINUED ON THE NEXT PAGE**

**Question 9**

When a particle is fired in the open from a point $O$ at a speed of $40\,ms^{-1}$ and at an acute angle $\theta$ above the horizontal, the equations of motion are:

$x = 40t\cos\theta$ and $y = 40t\sin\theta - 5t^2$ DO NOT PROVE THESE

The particle is fired with the same speed from a point $O$ on the floor of a horizontal tunnel of height $20m$.




Find the maximum horizontal range of the particle along the tunnel. **3**

**END OF EXAMINATION**

# STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \; if \; n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \; x > 0$

2014 EXTENSION 1 HSC TASK JUNE SOLUTIONS

1. $f(1.5) = 1.5^4 - 2 \times 1.5^2 + 1.5 - 3$
$= -0.9375$ ✓ calculates $f(1.5)$

$\therefore$ Root lies between $x = 1.5$ and $x = 2$

$\therefore$ Closer to 2 than 1. ✓

Sylvia isn't correct.

(Allow 1 mark if student answers 'not necessarily, she would need to find $f(1.5)$' without doing so.

2. $$\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$$

let $u = \sqrt{x}$

$du = \frac{1}{2} x^{-\frac{1}{2}} dx$

$du = \frac{dx}{2\sqrt{x}}$ ✓

$$= 2\int \frac{dx}{2\sqrt{x}(\sqrt{x}+1)}$$

$$= 2\int \frac{du}{u+1} \quad ✓$$

$$= 2\ln(u+1) + C$$

$$= 2\ln(\sqrt{x}+1) + C \quad ✓$$

3. $$\int_0^{\frac{\pi}{2}} \cos x \sqrt{\sin^5 x}\, dx$$

let $u = \sin x$

$du = \cos x\, dx$

when $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$

$x = 0$ $u = \sin 0 = 0$ ✓

$$= \int_0^1 u^{\frac{5}{2}}\, du \quad ✓$$

$$= \frac{2}{7}\left[u^{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{7}(1-0)$$

$$= \frac{2}{7} \quad ✓$$

4. $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $\qquad f(x) = \log x + \sin x$

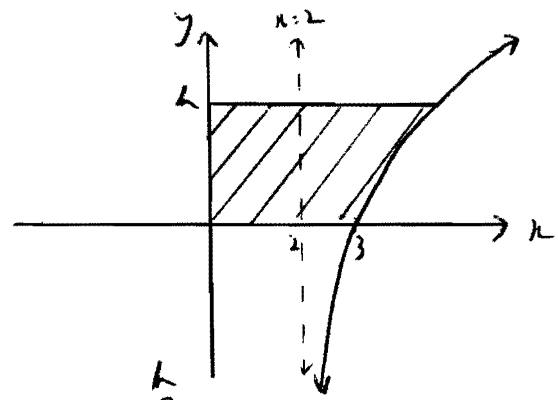
$f'(x) = \frac{1}{x} + \cos x$

$x_1 = 0.5 - \frac{(\ln 0.5 + \sin \frac{1}{2})}{2 + \cos 0.5}$ ✓

$= 0.57427\ldots$ ✓

$\therefore x_1 = 0.57$ (2dp) is a better approximation

5. a)




✓

b) $V = \pi \int_0^h x^2 \, dy$ $\qquad y = \log(x-2)$

$= \pi \int_0^h (e^y + 2)^2 \, dy$ ✓ $\qquad e^y = x - 2$

$x = e^y + 2$

$= \pi \int_0^h e^{2y} + 4e^y + 4 \, dy$

$= \pi \left[ \frac{e^{2y}}{2} + 4e^y + 4y \right]_0^h$ ✓

$= \pi \left( \left( \frac{e^{2h}}{2} + 4e^h + 4h \right) - \left( \frac{e^0}{2} + 4 + 0 \right) \right)$

$V = \pi \left( \frac{e^{2h}}{2} + 4e^h + 4h - \frac{9}{2} \right)$ ✓

c) $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$50 = \pi (e^{2h} + 4e^h + 4) \times \frac{dh}{dt}$ ✓

When $h = 1.5$cm

$\frac{dh}{dt} = \frac{50}{\pi(e^3 + 4e^{1.5} + 4)}$

$= 0.37882\ldots$

$\frac{dh}{dt} = 0.379$ cm/s (3dp) ✓

6 a) $v = (1-x)^2$

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$a = \frac{d}{dx}\left(\frac{1}{2}(1-x)^4\right)$ ✓

$a = 2(1-x)^3 \cdot -1$

$a = -2(1-x)^3$ ✓

b) $\frac{dx}{dt} = (1-x)^2$

$\frac{dt}{dx} = (1-x)^{-2}$

$\int_0^t dt = \int_0^x (1-x)^{-2}\,dx$

$\left[t\right]_0^t = \left[\frac{(1-x)^{-1}}{-1 \times -1}\right]_0^x$ ✓

$t = \left[\frac{1}{1-x}\right]_0^x$

$t = \frac{1}{1-x} - \frac{1}{1-0}$

$t+1 = \frac{1}{1-x}$

$1-x = \frac{1}{t+1}$

$x = 1 - \frac{1}{t+1}$ $\left(\text{or } x = \frac{t}{t+1}\right)$ ✓

c) Initial speed : when $x=0$ $v = (1-0)^2$

$= 1\,ms^{-1}$

1% of initial speed, $v = 0.01$

$v = \frac{dx}{dt}$

$v = (t+1)^{-2}$

$v = \frac{1}{(t+1)^2}$

when $v = 0.01$ $0.01 = \frac{1}{(t+1)^2}$

$(t+1)^2 = 100$

$t+1 = 10$ $\therefore t = 9$ seconds

7 a) $\ddot{y} = -10$

$\dot{y} = \int -10\,dt$

$\dot{y} = -10t + c$

when $t=0$, $\dot{y}=0$

$0 = 0 + c$

$c = 0$

$\therefore \dot{y} = -10t$

$y = \int -10t\,dt$

$y = -5t^2 + c_1$

when $t=0$, $y=3.2$

$3.2 = 0 + c_1$

$c_1 = 3.2$

$\therefore y = 3.2 - 5t^2$ ✓

(1) correct solution

b) when $y=0$  $0 = 3.2 - 5t^2$

$5t^2 = 3.2$

$t^2 = 0.64$

$t = 0.8$ ✓ $(t>0)$

c) $d = Vt$

$4 = V \times 0.8$

$V = 5$

$\therefore$ Cat should jump at $5\,ms^{-1}$ ✓

8 a)

$x = -a\sin(3t+\alpha)$ (i)

$\dot{x} = -3a\cos(3t+\alpha)$

$\ddot{x} = 9a\sin(3t+\alpha)$

$\ddot{x} = -9\times -a\sin(3t+\alpha)$

$\ddot{x} = -9x$ from (i)

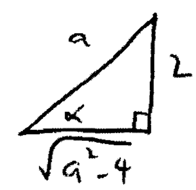
b) when $t=0$, $\ddot{x} = 18$

$18 = 9a\sin\alpha$

$\sin\alpha = \frac{2}{a}$

when $t = \frac{5\pi}{18}$, $v = 12$

$12 = -3a\cos\left(\frac{15\pi}{18} + \alpha\right)$




$-4 = a\left(\cos\frac{15\pi}{18}\cos\alpha - \sin\frac{15\pi}{18}\sin\alpha\right)$

$-4 = \cancel{a}\left(-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{a^2-4}}{\cancel{a}} - \frac{1}{2}\cdot\frac{2}{\cancel{a}}\right)$

$-4 = \frac{-\sqrt{3}\sqrt{a^2-4}}{2} - 2$

$-8 = -\sqrt{3(a^2-4)} - 2$

$-6 = -\sqrt{3(a^2-4)}$

$36 = 3(a^2-4)$

$a^2-4 = 12$

$a^2 = 16$

$a = 4 \quad (a>0)$

c) max velocity at centre of in SHM ie $x=0$

$\dot{x} = -12\cos(3t+\alpha)$

$\therefore$ max velocity is $12\ ms^{-1}$ as $-1 \le \cos(3t+\alpha) \le 1$.

d) $\sin\alpha = \frac{2}{4}$ from (b)

$\therefore \alpha = 30°$

$\therefore \dot{x} = -12\cos(3t+\frac{\pi}{6})$

changes direction when $\dot{x}=0$ since in SHM

$\therefore 0 = \cos(3t+30°)$

$3t+\frac{\pi}{6} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \ldots$

$3t = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}, \frac{13\pi}{3}$ ← greater than 4 seconds.

$t = \frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \cancel{\frac{13\pi}{9}}$

$\therefore$ changes direction 4 times.

8(b) Alternative

Q8

If amplitude is 4m

$x = -4\sin(3t+\alpha)$

$\dot{x} = -12\cos(3t+\alpha)$

$\ddot{x} = 36\sin(3t+\alpha)$

But $\ddot{x} = 18$ when $t=0$

$18 = 36\sin\alpha$

$\sin\alpha = \frac{1}{2}$

$\alpha = 30^\circ$

check in $\dot{x}$: when $t = \frac{5\pi}{18}$, $\dot{x} = 12$

$\dot{x} = -12\cos\left(\frac{15\pi}{18} + 30^\circ\right)$

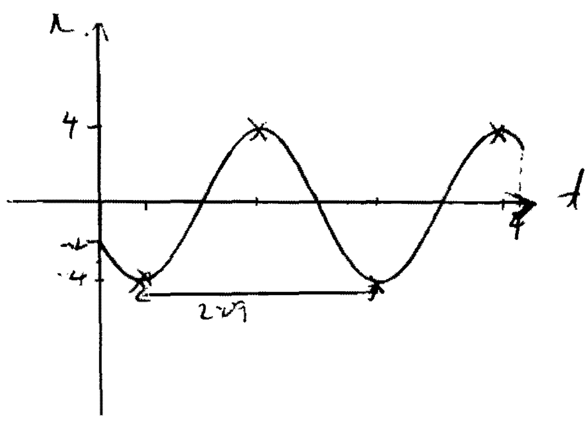
$\dot{x} = -12\cos\left(\frac{5\pi}{6} + \frac{\pi}{6}\right)$

$\dot{x} = -12\cos\pi$

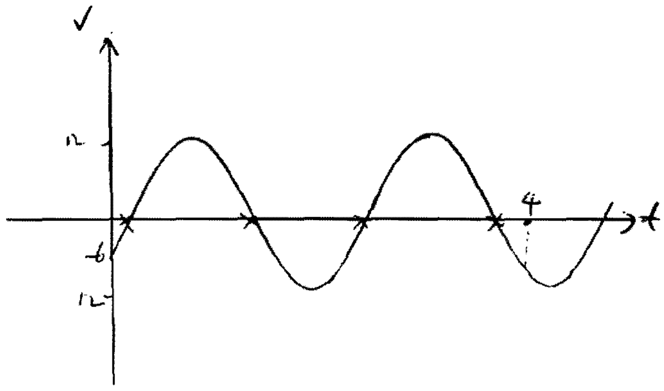
$\therefore \dot{x} = 12$ as req'd.

$\therefore a = 4$.

8(d) OR graphically




4 changes




4 changes

9. Max range will occur if top of flight path is at a height of 20m.

$y = 40t\sin\theta - 5t^2$

$\dot{y} = 40\sin\theta - 10t$

when $\dot{y} = 0$ max height is reached

$0 = 40\sin\theta - 10t$

$10t = 40\sin\theta$

$t = 4\sin\theta$ ✓

For max range $y = 20$.

$20 = 10t(4\sin\theta) - 5t^2$

$20 = 10t^2 - 5t^2$

$5t^2 = 20$

$t^2 = 4$

$t = 2$ $\therefore$ Time of flight = 4 seconds ✓

when $t = 2$, $2 = 4\sin\theta$

$\sin\theta = \frac{1}{2}$

$\theta = 30°$

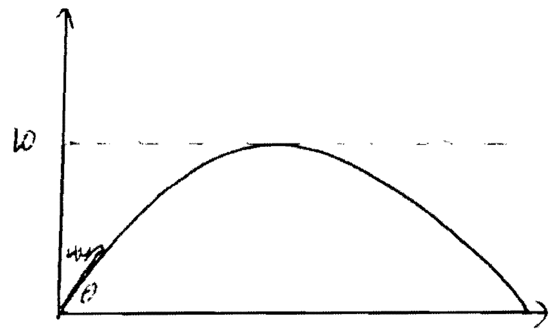
$\therefore$ max range $= 40 \times 4 \times \cos 30°$

$= 160\cos 30$

$= 80\sqrt{3}$ m. ✓

Method 2

9.




$x = 40t\cos\theta \qquad y = 40t\sin\theta - 5t^2$

$\therefore\ t = \dfrac{x}{40\cos\theta}$ sub in $y \rightarrow$ $y = \dfrac{40x\sin\theta}{40\cos\theta} - \dfrac{5x^2}{1600\cos^2\theta}$

$$y = x\tan\theta - \frac{x^2}{320\cos^2\theta} \quad (i)$$

when $y=20$ $\quad 20 = x\tan\theta - \dfrac{x^2}{320\cos^2\theta}$

$$\frac{x^2}{320\cos^2\theta} - x\tan\theta + 20 = 0$$

For max range $\Delta = 0$

$$0 = \tan^2\theta - \frac{4\times 20}{320\cos^2\theta}$$

$$0 = \sin^2\theta - \frac{1}{4}$$

$\sin^2\theta = 1/4$

$\sin\theta = 1/2$

$\theta = 30^\circ$

For max range $y=0$ and $\theta = 30^\circ$ in (i)

$$0 = x\tan 30^\circ - \frac{x^2}{320\cos^2 30^\circ}$$

$$0 = \frac{x}{\sqrt{3}} - \frac{x^2}{320\times\frac{3}{4}}$$

$$0 = x\left(\frac{1}{\sqrt{3}} - \frac{x}{240}\right)$$

$x = 0$ or $\quad \dfrac{1}{\sqrt{3}} - \dfrac{x}{240} = 0$

starting point

$$\frac{x}{240} = \frac{1}{\sqrt{3}}$$

$x = \dfrac{240}{\sqrt{3}}$ or $\dfrac{240\sqrt{3}}{3} = \underline{\underline{80\sqrt{3}\text{ m}}}$