BAULKHAM HILLS HIGH SCHOOL

2016

YEAR 12 TASK 3
TERM 2 ASSESSMENTS

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- Show all necessary working in Questions 6-9
- Marks may be deducted for careless or badly arranged work

Total marks – 35

Exam consists of 8 pages and 4 Questions

Reference Sheet is on page 7 and 8.

Answer each question on the appropriate page.
All necessary working should be shown in every question.

Question 1 (10 marks) **Marks**

a) Use the substitution $u = x^2 + 1$ to find the indefinite integral **3**

$$I = \int \frac{x\,dx}{\sqrt{x^2+1}}$$

b) Use the substitution $x = 3\sin\theta$ to find the integral **3**

$$\int_{\sqrt{3}}^{3} \sqrt{9 - x^2}\,dx$$

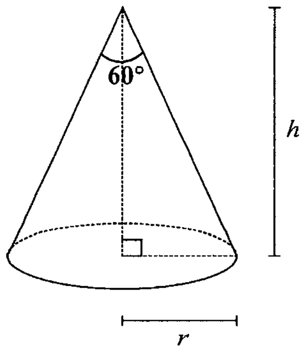
c) (i) Show that $\tan^{-1} x - x^2 + \frac{\pi}{4} = 0$ has a root in the interval $1 < x < \sqrt{3}$ **2**

(ii) Use one application of Newton's method to find an approximation to this root. Take $x = 1.5$ as the first approximation expressing your answer to one decimal place. **2**

End of Question 1

Question 2 (9 marks) **Marks**

a) Grain is poured at a constant rate of 0.25 cubic metres per second. It forms a conical pile, with the angle at the apex of the cone equal to $60°$. The height of the pile is h metres, and the radius of the base is r metres.



(i) Show that $r = \frac{h}{\sqrt{3}}$ **1**

(ii) Show that V, the volume of the pile is given by $V = \frac{\pi h^3}{9}$ **1**

(iii) Hence find the rate at which the height of the pile is increasing when the height of the pile is 3 metres. **2**

b) The acceleration of the particle P is given by the equation

$$\frac{d^2x}{dt^2} = 8x - 2x^3.$$

where x metres is the displacement of P from a fixed point O after t seconds. Initially, when $x = 2$ the velocity of the particle is $3\ \text{ms}^{-1}$.

(i) Find the equation for v^2. **3**

(ii) Between which two points the particle is moving. **2**

End of Question 2

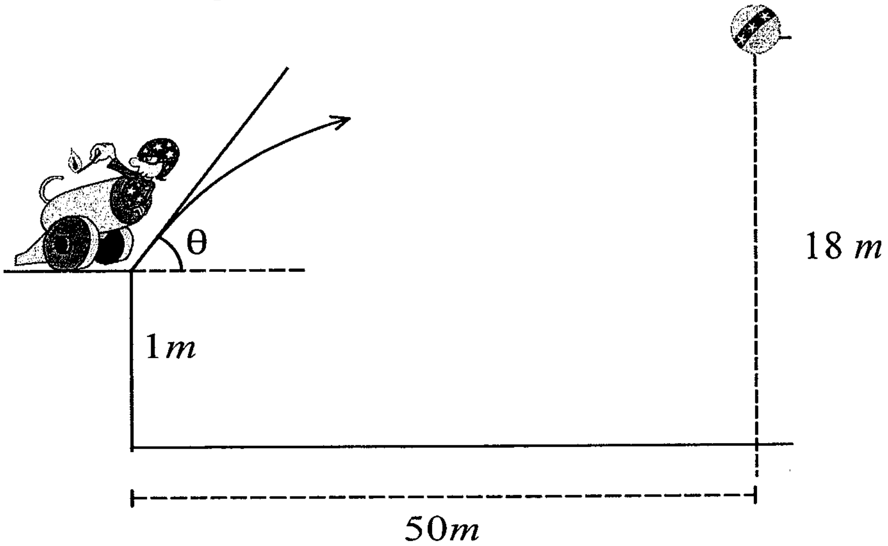
Question 3 (9 marks)

a) A particle is moving along the x-axis, starting from a position 2 metres to the right of the origin with an initial velocity of 5 ms^{-1} and an acceleration given by $\ddot{x} = 2x^3 + 2x$.

(i) Show that $\dot{x} = x^2 + 1$. **3**

(ii) Hence find an expression for t in terms of x. **2**

b) A circus act consists of a person fired from a cannon at 30 metres per second on a platform 1 metre high. The moment the cannon is fired, a ball will be released 50 metres away, from a height of 18metres. Let $g = 10ms^{-2}$.



The equations of motion of the projectile of the person are :

$x = 30t\cos\theta$ DO NOT PROVE THESE

$y = 1 + 30t\sin\theta - 5t^2$

The equations of motion of the ball are:

$x = 50$ DO NOT PROVE THESE

$y = 18 - 5t^2$

(i) What is the angle at which the cannon must be fired so the person collides with the ball? **2**

(ii) At what height does the person collides with the ball? **2**

End of Question 3

Question 4 (7 marks) **Marks**

a) A particle moving in a straight line is performing Simple Harmonic Motion. At time t seconds it has displacement x metres, where $x = 4\cos^2 t - 2\sin^2 t$.

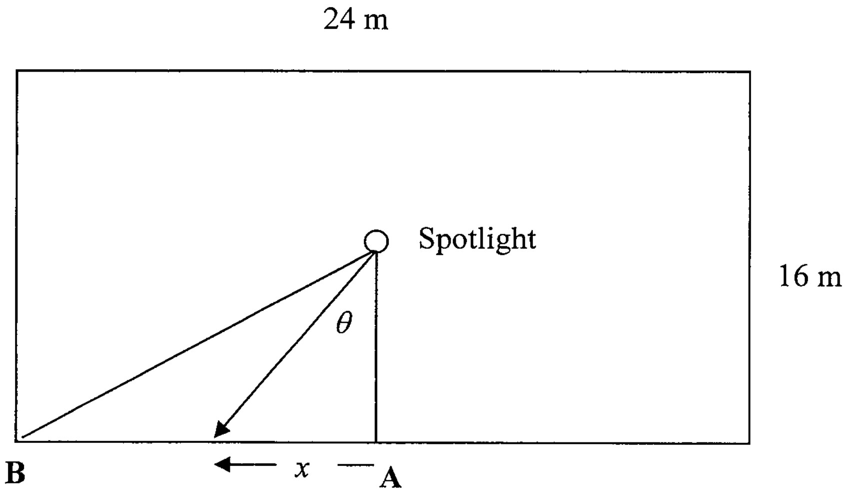
(i) Show that $x = 1 + 3\cos 2t$. **1**

(ii) Hence express the acceleration $\ddot{x}\text{ms}^{-1}$ of the particle in the form of $\ddot{x} = -n^2(x-b)$ **1**

(iii) Which points is the particle oscillating? **1**

(iv) At which point is the particle moving the fastest? **1**

b) A spotlight is in the centre of a rectangular nightclub which measures 24 m by 16 m. It is spinning at a rate $\frac{d\theta}{dt} = 2\pi$ radians per second. Its beam projects a spot which moves along the walls as it spins.



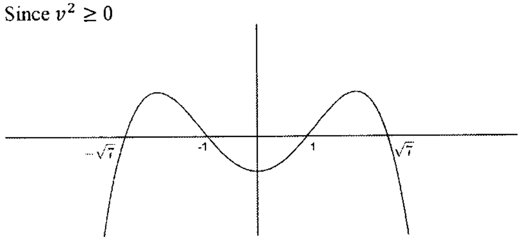
i) Find an expression for the velocity $\frac{dx}{dt}$ in terms of x at which the spot appears to be moving along the wall from A to B. **2**

ii) What is the velocity at which the spot appears to be moving at the point A? **1**

End of Exam

BAULKHAM HILLS HIGH SCHOOL
YEAR 12 EXTENSION 1 ~~TASK~~ Term 2 2016 SOLUTIONS

	Solution	Marks	Comments
1a)	$u = x^2 + 1$ $du = 2x\,dx$ $I = \frac{1}{2}\int \frac{du}{\sqrt{u}}$ $I = \frac{1}{2}\int u^{-\frac{1}{2}}\,du$ $I = \frac{1}{2} \times 2 \times u^{\frac{1}{2}} + c$ $I = \sqrt{x^2+1} + c$	3	***3 marks*** • *Correct Solution* ***2 marks*** • *Integrates correctly and leaves in terms of u* • *Significant progress towards the solution.* ***1 mark*** • *Correct expression for integral in terms of u*
1b)	$x = 3\sin\theta$ $dx = 3\cos\theta\,d\theta$ *if* $x = 3$, $3 = 3\sin\theta$, $\theta = \frac{\pi}{2}$ *if* $x = \sqrt{3}$, $\sqrt{3} = 3\sin\theta$, $\theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ $I = \int_{\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)}^{\frac{\pi}{2}} \sqrt{9 - 9\sin^2\theta} \times 3\cos\theta\,d\theta$ $I = \int_{\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)}^{\frac{\pi}{2}} 3\cos\theta \times 3\cos\theta\,d\theta$ $I = 9\int_{\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)}^{\frac{\pi}{2}} \frac{1}{2}(\cos 2\theta + 1)\,d\theta$ $I = \frac{9}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_{\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)}^{\frac{\pi}{2}}$ $I = \frac{9}{2}\left[\left(\frac{1}{2}\sin\pi + \frac{\pi}{2}\right) - \left(\frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) + \frac{\pi}{6}\right)\right]$ $I = 2.18\,(2dp)$	3	***3 marks*** • *Correct Solution* ***2 marks*** • *Correct integral in terms of dθ (incl. limits)* ***1 mark*** • *Correct integral in terms of dθ*
1(i)	Let $f(x) = \tan^{-1}x - x^2 + \frac{\pi}{4}$ $f(1) = \tan^{-1}1 - 1^2 + \frac{\pi}{4}$ $f(1) = 0.57 > 0$ $f(\sqrt{3}) = \tan^{-1}\sqrt{3} - \sqrt{3}^2 + \frac{\pi}{4}$ $f(\sqrt{3}) = -1.16 < 0$ Since $f(1) > 0$, $f(\sqrt{3}) < 0$ and $f(x)$ is continuous, then there is at least one root between $x = 1$ and $x = \sqrt{3}$	2	***2 marks*** • *Correct Solution* ***1 mark*** • *States f(x) is continuous* • *shows f(1) > 0 and f(√3) < 0*
1c(ii)	$f'(x) = \frac{1}{x^2+1} - 2x$ $f'(1.5) = \frac{1}{1.5^2+1} - 2(1.5) = -2.692$ $x_1 = x_0 - \frac{f(1.5)}{f'(1.5)}$ $x_1 = 1 - \frac{0.4818}{-2.692} = 1.1127\ldots$ $x_1 = 1.3$	2	***2 marks*** • *Correct Solution* ***1 mark*** • *Correct use of formula with wrong values* • *finding f′(1.5)*
2a(i)	In ΔOPA, $\angle OPA = \frac{1}{2} \times 60° = 30°$ $\tan 30° = \frac{r}{h}$ $r = h\tan 30°$ $r = \frac{h}{\sqrt{3}}$	1	***1 mark*** • *Correct Solution*
2a(ii)	Volume of the cone $= \frac{1}{3}\pi r^2 h$ $V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h$ $V = \frac{\pi h^3}{9}$	1	***1 mark*** • *Correct Solution*
2a(iii)	$\frac{dV}{dt} = 0.25$ $\frac{dV}{dh} = \frac{\pi h^2}{3}$ When $h = 3$, $\frac{dV}{dh} = 3\pi$ $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $0.25 = 3\pi \times \frac{dh}{dt}$ $\therefore \frac{dh}{dt} = \frac{1}{12\pi}$ ∴ the height of the pile is increasing at a rate of $\frac{1}{12\pi}ms^{-1}$ when $h = 3$	2	***2 marks*** • *Correct Solution* ***1 mark*** • *Finds a correct expression using* $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$
2b(i)	$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8x - 2x^3$ $\frac{1}{2}v^2 = 4x^2 - \frac{1}{2}x^4 + c$ $v^2 = 8x^2 - x^4 + 2c$ When $v = 3, x = 2$ $3^2 = 8(2)^2 - (2)^4 + 2c$ $7 = 2c$ $v^2 = 8x^2 - x^4 + 7$	2	***2 marks*** • *Correct Solution* ***1 mark*** • *Uses* $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

	Solution	Marks	Comments
2b(ii)	$v^2 = (7 - x^2)(1 - x^2)$ $v^2 = (\sqrt{7} - x)(\sqrt{7} + x)(1 - x)(1 + x)$ Since $v^2 \geq 0$  $-\sqrt{7} \leq x \leq -1$ or $1 \leq x \leq \sqrt{7}$ Since it initially starts at $x = 2$ $\therefore 1 \leq x \leq \sqrt{7}$ (Note A better students solution would have explained: that at $x = 2, v = 3$. As $x = \sqrt{7}, a < 0$ (ie. gradient of $v < 0$) so the particle moves left. As $x = 1,\ a > 0$ (ie. gradient of $v > 0$) so the particle moves right. And the particle will oscillate between $1 \leq x \leq \sqrt{7}$ and the motion is not in SHM.)	2	***2 marks*** • *Correct Solution* ***1 mark*** • *Finds all values of* $v = 0$
3a(i)	$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 2x$ $\frac{1}{2}v^2 = \frac{1}{2}x^4 + x^2 + c$ When $t = 0, x = 2, v = 5$ $\frac{1}{2}(5)^2 = \frac{1}{2}(2)^4 + (2)^2 + c$ $c = \frac{1}{2}$ $v^2 = x^4 + 2x^2 + 1$ $v = \pm\sqrt{(x^2 + 1)^2}$ When $x = 2, v = 5$ $v = x^2 + 1$	3	***3 marks*** • *Correct solution by considering* $\pm$cases ***2 marks*** • *Finding a correct expression for v* ***1 mark*** • *Finding c*
3a(ii)	$\frac{dx}{dt} = x^2 + 1$ $\frac{dt}{dx} = \frac{1}{x^2 + 1}$ $t = \tan^{-1} x + c$ When $t = 0, x = 2$ $c = -\tan^{-1} 2$ $t = \tan^{-1} x - \tan^{-1} 2$	2	***2 marks*** • *Correct Solution* ***1 mark*** • *Correct expression for* $\frac{dt}{dx}$
3b(i)	$x = 30t\cos\theta$ $x = 20$ $y = 1 + 30t\sin\theta - 5t^2$ $y = 18 - 5t^2$ To intercept, the x and y values must be equal. $30t\cos\theta = 20$ $t\cos\theta = \frac{5}{3}$ ... (1) $1 + 30t\sin\theta - 5t^2 = 18 - 5t^2$ $t\sin\theta = \frac{17}{30}$ ... (2) $\frac{(1)}{(2)} = \frac{t\sin\theta}{t\cos\theta} = \frac{5}{3} \div \frac{17}{30}$ $\tan\theta = \frac{17}{50}$ $\theta = 18°47'$ (to the nearest minute)	2	***2 marks*** • *Correct Solution* ***1 mark*** • *Equates the x and y equations* • *Equates one equation to find* θ
3b(ii)	$t = \frac{17}{30\sin 18°47'}$ $t = 1.76\ sec$ $y = 18 - 5(1.76)^2$ $y = 2.51m$	2	***2 marks*** • *Correct Solution* ***1 marks*** • *Finds t*
4a(i)	$x = 1 + 3\cos 2t$ $x = \cos^2 x + \sin^2 x + 3(\cos^2 t - \sin^2 t)$ $x = 4\cos^2 x - 2\sin^2 t$	1	***1 marks*** • *Correct Solution*
4a(ii)	$\dot{x} = -6\sin 2t$ $\ddot{x} = -12\cos 2t$ $\ddot{x} = -4(3\cos 2t)$ $\ddot{x} = -4(x - 1)$ Where $n = 2$ and $b = 1$	1	***1 marks*** • *Correct Solution*
4a(iii)	the particle oscillates between $x = -2$ and $x = 4$ $ie. -2 \leq x \leq 4$	1	***1 marks*** • *Correct Solution*
4a(iv)	$x = 1 + 3\cos 2t$ The particle moves fastest at the centre of motion at $x = 1$	1	***1 mark*** • *Correct Solution*

	Solution	Marks	Comments
4b(i)	$\tan\theta = \frac{x}{8}$ $x = 8\tan\theta$ $\frac{dx}{d\theta} = 8\sec^2\theta$ $\frac{dx}{dt} = \frac{dx}{d\theta}.\frac{d\theta}{dt}$ $= 8\sec^2\theta \times 2\pi$ $= 16\pi\sec^2\theta$ $= 16\pi\left(1+\tan^2\theta\right)$ $= 16\pi\left(1+\tan^2\theta\right)$ $= 16\pi\left(1+\left(\frac{x}{8}\right)^2\right)$ $= 16\pi + \frac{\pi^2 x}{4}$	2	***2 marks*** • *Correct Solution* ***1 mark*** • *Finds* $\frac{dx}{d\theta}$ • *Correct use of the chain rule with substitution.*
4b(ii)	At A, $x = 0$ $\frac{dx}{dt} = 16\pi\left(1+\left(\frac{0}{8}\right)^2\right)$ $= 16\pi$	1	***1 mark*** • *Correct Solution*