BAULKHAM HILLS HS PERSEVERE

Year 12
HSC Assessment Task
June 2017

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using non erasable black or blue pen
- Board-approved calculators may be used
- Show all necessary working
- Marks may be deducted for careless or badly arranged work
- Reference sheet is provided at the back of this paper.

Total marks – 48

Questions 1-3 (pages 2-4)

Answer each question on the appropriate pages of your answer booklet.

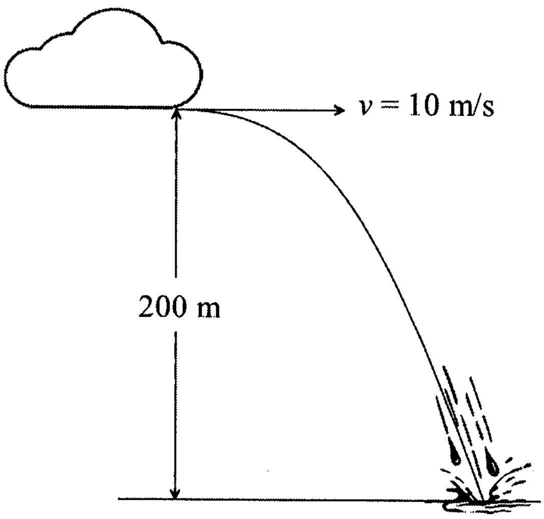
Question 1 (17 marks) - Use the Question 1 section of the writing booklet.		Marks
a)	Find $\int \frac{\sin x}{1+\cos x} dx$ using the substitution $u = \cos x$.	3
b)	Find $\int \frac{dx}{x(1+\ln x)^5}$ using the substitution $u = 1 + \ln x$.	3
c)	Find the exact value of $\int_{-1}^{2} \frac{dx}{x^2+2x+10}$ using the substitution $u = x + 1$	3
d)	Given that $x = 0.7$ is an approximate root of the equation $\cos 2x = x$, use one application of Newton's method to obtain another approximation to this root. Give your answer correct to two decimal places.	3
e)	Given $f(x) = x - 4 + e^{2x}$. (i) Show that there is a zero between $x = 0.6$ and $x = 0.7$. (ii) Use one application of halving the interval method to find a smaller interval containing the zero. (iii) Hence solve $x = 4 - e^{2x}$ correct to one decimal place, given that $x = 4 - e^{2x}$ has only one solution.	 2 2 1
	End of Question 1	

Question 2 **(16 marks) - Use the Question 2 section of the writing booklet.** **Marks**

a) A particle is moving so that its velocity at certain position x metres is given by $v^2 = 400 - 2x$.
Particle is initially at the origin and travelling with a velocity of 20 metres per second.

(i) Find x in terms of time t. **3**
(ii) Find when the particle is stationary. **2**
(iii) Find the time when the particle returns back to the origin. **2**
(iv) Describe the motion after the particle returns back to the origin **1**

b) A steady wind is blowing with speed of 10 metres per second. From clouds moving horizontally with the wind, heavy raindrops fall to the ground 200 metres below. Air resistance is neglected and the approximate value of gravity g is 10 m/s².



(i) Derive the equation for the horizontal (x) and vertical (y) displacement of a raindrop. **3**
(ii) Find the time when the drop hits the ground. **2**
(iii) Find the speed and acute angle at which the raindrop hits the ground. **3**

End of Question 2

Question 3 (15marks) - Use the Question 3 section of the writing booklet.		Marks
a)	A particle moves along the x-axis such that at time t seconds, the acceleration $\ddot{x}$ is given by $\ddot{x} = \frac{-1}{4(x-2)^3}$, where x is the displacement in metres from the origin. Find the expression of velocity v in terms of x, if initially the particle is 3 metres to the right from the origin travelling with the velocity $v = 0.5m/s$.	**3**
b)	A particle is moving in Simple Harmonic Motion, so that its displacement x centimetres from the origin at time t seconds is given by $x = 1 + 4\cos(3t) + 4\sin(3t)$. (i) Express $\ddot{x}$ in the form $\ddot{x} = -n^2(x - x_0)$ where x_0 is the centre of the motion. (ii) Find the period and amplitude of the motion. Answer in exact form. (iii) Find the maximum distance of the particle from the origin. (iv) Find the maximum acceleration of the particle. (v) Find the speed of the particle when $x = 2$ cm.	 **2** **2** **1** **1** **2**
c)	Michelle is standing at a point M watching a balloon being released from a point W which is 100 metres away on the horizontal ground. The balloon rises vertically at a constant velocity of 5 metres per second. Let θ radians be the angle of elevation of the balloon at time t seconds and x metres be the distance the balloon has travelled in that time. (i) Find the expression for the rate of change of the angle of elevation of the balloon over time. Answer in terms of x. (ii) Find the rate of change of the angle of elevation over time of the balloon when $\theta = \frac{\pi}{4}$.	 **3** **1**
	End of the Exam	

Yr. 12 - HSC Ass. 3
June 2017

Question 1

	Marks
a) $\int \frac{\sin x}{1+\cos x}\,dx$ $u=\cos x$, $du=-\sin x\,dx$	3 - correct solns.
	2 - obtains correct integrand in terms of u-only
$= \int \frac{-du}{1+u}$	1 - correct subst.
$= -\ln\lvert 1+u\rvert + c = -\ln\lvert 1+\cos x\rvert + c$	1 - finding du correctly
	1 - correctly integrates
b) $\int \frac{dx}{x(1+\ln x)^5}$ $u=1+\ln x$, $du=\frac{1}{x}dx$	3 - correct solns.
	2 - obtains correct integrand in terms of u-only
$= \int \frac{du}{u^5}$	1 - correct subst.
$= \frac{u^{-4}}{-4} + C = -\frac{1}{4(1+\ln x)^4} + C$	1 - finding du correctly
	1 - integrates correctly
c) $\int_{-1}^{2} \frac{dx}{x^2+2x+10}$ $u=x+1$, $u^2=x^2+2x+1$, $du=dx$	3 - correct solns.
	2 - correct integrand in terms of u only with changed limits
$= \int_0^3 \frac{du}{u^2+9}$ $x=-1 \therefore u=0$, $x=2 \therefore u=3$	
	1 - correct integrand in terms of u without changed limits
$= \frac{1}{3}\left[\tan^{-1}\frac{u}{3}\right]_0^3$	
	1 - integrades correctly
$= \frac{1}{3}\left[\tan^{-1}1 - \tan^{-1}0\right] = \frac{1}{3}\times\frac{\pi}{4} = \frac{\pi}{12}$	

	Marks
d) $\cos 2x = x$:	
$\cos 2x - x = 0$ let $f(x)=\cos 2x - x$	3 - correct solns.
$f'(x) = -2\sin 2x - 1$	2 - finds a relevant function, its derivative & applies Newton's method correctly
if $x_0 = 0.7$	
$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$	1 - finds a relevant function
$x_1 = 0.7 - \frac{\cos(2\times0.7) - 0.7}{-2\sin(2\times0.7) - 1}$	1 - derives $f(x)$ correctly
	1 - applies Newton's method correctly
$x_1 = 0.52159\ldots$	
$\therefore x_1 = 0.52$ (2 d.p)	
e) i) $f(0.6) = -0.0798.. < 0$	2 - correct solns.
$f(0.7) = 0.7551\ldots > 0$	1 - finds $f(0.6)$ and $f(0.7)$ correctly
since $f(0.6) < 0$ and $f(0.7) > 0$	
and $f(x)$ is continuous	
$\therefore 0.6 < x < 0.7$ [x = zero]	
ii) $f\left(\frac{0.6+0.7}{2}\right) = f(0.65) = 0.31929..$	2 - correct solns.
$\therefore 0.6 < x < 0.65$	1 - applies the method correctly
iii) $f(x)=0 \therefore 0 = x - 4 + e^{2x}$	1 - correct solns.
$x = 4 - e^{2x}$	
from (ii) $0.6 < x < 0.65$	0 - no mark for bald answer of $x=0.6$.
$\therefore x = 0.6$ (1 d.p)	

Question 2

	Marks
a) Method ①	3 - correct solns.
i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}(200 - x)$	2 - derives $\ddot{x} = -1$ and uses $\dot{x} = \int \ddot{x}\,dt$ to find v in terms of t
$\therefore \ddot{x} = -1 \quad (m/s^2)$	
now $\dot{x} = \int \ddot{x}\,dt = \int -1\,dt = -t + C$	
$t = 0$, $\dot{x} = v = +20$: $20 = 0 + C \quad \therefore C = 20$	
$\therefore \dot{x} = -t + 20$	2 - for relevant method to find $\frac{dx}{dt}$
$x = \int \dot{x}\,dt = \int -t + 20\,dt$	
$x = -\frac{t^2}{2} + 20t + C$	
$t = 0$, $x = 0 \quad \therefore C = 0 \quad \boxed{x = \frac{-t^2}{2} + 20t}$	2 - obtains $t = f(x)$
	1 - finds $\ddot{x} = -1$
or relevant Method ②	1 - applies rules $v = \frac{dx}{dt}$, $\frac{dt}{dx} = \frac{1}{v}$
$v^2 = 400 - 2x$	
$\therefore v = \pm\sqrt{400 - 2x}$	$t = 0$, $\ddot{x} = -1$, $v = 0$; $x = 0$, $v = +20$ $\longrightarrow$ $x = 200$
$\therefore v = (+)\sqrt{400 - 2x}$ ①	1 - attempts to find t in terms of x
for $0 < x < 200$	$400 - 2x \geq 0$; $200 \geq x$
$\therefore \frac{dx}{dt} = \sqrt{400 - 2x} = (400 - 2x)^{\frac{1}{2}}$	1 - significant progress towards solution
$\therefore \frac{dt}{dx} = (400 - 2x)^{-\frac{1}{2}}$	
$t = \int (400 - 2x)^{-\frac{1}{2}}\,dx$	

$\therefore t = \frac{(400 - 2x)^{\frac{1}{2}}}{\frac{1}{2} \times -2} + C \qquad t = 0 \quad x = 0 \qquad C = 20$

$\therefore t = 20 - (400 - 2x)^{\frac{1}{2}}$ ①

$(400 - 2x)^{\frac{1}{2}} = 20 - t$

$400 - 2x = (20 - t)^2$

$400 - (20 - t)^2 = 2x$ ①

$\boxed{200 - \frac{1}{2}(20 - t)^2 = x}$ or $\boxed{x = \frac{-t^2}{2} + 20t}$

	Marks
ii) $v = 0$	2 - correct solns.
By using Method ① $\quad \dot{x} = v = -t + 20$; $t = 20$	1 - finds x when $v = 0$
By using Method ② $\quad v^2 = 400 - 2x$; $0 = 400 - 2x$; $x = 200$ ①	
sub. $x = 200 \therefore 200 = \frac{-t^2}{2} + 20t$ ... $t = 20$	
or sub. in $t = 20 - (400 - 2x)^{\frac{1}{2}}$ ... $t = 20$ ①	

Marks

iii) $x = 0$

$0 = -\frac{t^2}{2} + 20t$ (1)

$0 = t\left(-\frac{t}{2} + 20\right)$

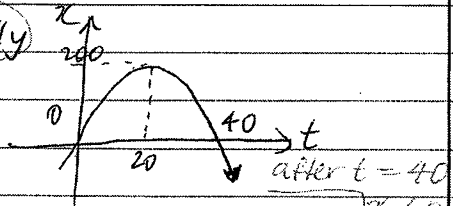
$t = 0$ initially $\qquad t = 40$

$\therefore$ particle returns to origin ~~after~~ at 40 seconds. (1)

2-correct solu

1-attempts to solve eqn. $x = 0$

iv) graphically



after $t = 40$ $x < 0$ & slope is negative $\therefore V < 0$ and $V \neq 0$ & slope is getting steeper $\therefore$ speeding up

1 - correct ansr

OR after returning to origin particle moves to the left since at $t = 40$ $V = -t + 20 = -20 < 0$ then after $t > 40$ $\therefore V < 0$ and $\ddot{x} = -1$ } keeps moving to the left speeding up. (1)

OR when $t > 40$

~~$x = 200 - \frac{(20 - t^2)}{2}$~~ < 0

$\therefore$ moves to left

and since V is never zero $\therefore$ keeps moving to the left at increasing speed

Marks

(b) i) $\ddot{x} = 0$ $\qquad t = 0 \quad v = 10$

$\dot{x} = \int 0\,dt = c$ $\qquad$ since $\theta = 0$

$\boxed{\dot{x} = 10}$ $\qquad \therefore \dot{x} = v\cos 0 = v$

$x = \int 10\,dt = 10t + c$ $\qquad t = 0 \; x = 0 \therefore c = 0$

$\therefore \boxed{x = 10t}$ (1)

$\ddot{y} = -g$

$\dot{y} = \int -g\,dt = -gt + c$ $\qquad t = 0$ $\quad \dot{y} = V\sin 0 = 0$ $\quad \therefore c = 0$

$\boxed{\dot{y} = -gt} = \boxed{-10t}$ (1) $\qquad g = 10$

$y = \int -10t\,dt$

$y = -5t^2 + c$ $\qquad t = 0 \; y = 200 = c$

$\boxed{y = -5t^2 + 200}$ (1)

3-correct solns

2 - correctly derive eqn. for x or

1-correctly start with $\ddot{x} = 0$ and $\ddot{y} = -g$ integrating to obtain $\dot{x}, \dot{y}$

ii) hits the ground $\therefore y = 0$

$\therefore 0 = -5t^2 + 200$ (1)

$t^2 = 40$ & $t > 0$ $\therefore t = \sqrt{40}$

$\therefore \boxed{t = 2\sqrt{10}}$ (1)

2-correct solns.

1-attempts to solve eqn. for $y = 0$

iii) at $t = 2\sqrt{10}$ $\quad \dot{x} = 10$

$\dot{y} = -10t =$ (1)

$= -10 \times 2\sqrt{10}$

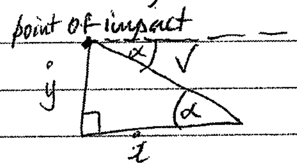
$\dot{y} = -20\sqrt{10}$

You may ask for extra writing paper if you need more space to answer question 12

Marks

Question 2b) cont.

iii)



$\dot{x} = 10$

$\dot{y} = -20\sqrt{10}$

3 – correct solns.

speed $= |v| = \left| \pm\sqrt{\dot{y}^2 + \dot{x}^2} \right|$

1 – relates $\dot{x}$ & $\dot{y}$ correctly

speed $= \sqrt{(-20\sqrt{10})^2 + 10^2} = \sqrt{4100} = 10\sqrt{41}$

1 – finds speed

or

$\therefore$ speed $\doteqdot 64.03$ m/s

OR (1)

1 – finds acute angle

$\tan\alpha = \left|\frac{\dot{y}}{\dot{x}}\right| = \left|\frac{-20\sqrt{10}}{10}\right|$ α – acute

$\alpha = \tan^{-1} 2\sqrt{10}$

$\boxed{\alpha = 81^\circ\, 1'}$ (1)

Question 3

Marks

a) $\ddot{x} = \frac{-1}{4(x-2)^3}$

3 – correct solns.

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{4}(x-2)^{-3}$

2 – finds correct expression for v^2

$\frac{1}{2}v^2 = -\frac{1}{4}\int (x-2)^{-3}\,dx$

1 – applies formula $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ correctly

$\frac{1}{2}v^2 = -\frac{1}{4}\,\frac{(x-2)^{-2}}{-2} + c$

$\frac{1}{2}v^2 = \frac{1}{8}(x-2)^{-2} + c$ (1)

1 – correctly integrates & finds constant

$x = +3 \quad v = 0.5$

$\therefore \frac{1}{2}(0.5)^2 = \frac{1}{8}(3-2)^{-2} + c \quad \therefore c = 0$

$\therefore \frac{1}{2}v^2 = \frac{1}{8}(x-2)^{-2}$

1 – correctly decides about the sign for velocity

$\therefore v^2 = \frac{1}{4(x-2)^2}$ (1)

$\therefore v = \pm\frac{1}{2(x-2)}$

but $t=0 \quad x=3 \quad v=+0.5$

$\therefore v = (+)\frac{1}{2(x-2)}$ (1)

OR $t=0$, $x=3$ $v = \frac{1}{2(x-2)} \neq 0$

$v = +0.5$ keeps moving to left

$\ddot{x} = -1/4 < 0$

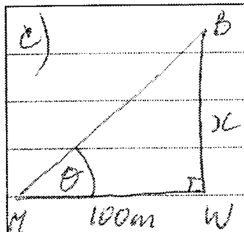
and for $x>3$ $v>0$ & $\ddot{x}<0$ slowing down $\therefore v \to 0$

	Marks
b) i) $x = 1 + 4\cos(3t) + 4\sin(3t)$	2 - correct solns.
$\dot{x} = -4\times 3\sin(3t) + 12\cos(3t)$	
$\ddot{x} = -36\cos(3t) - 36\sin(3t)$	1 - differentiates x & $\dot{x}$ correctly
$\therefore \ddot{x} = -9\left(4\cos(3t) + 4\sin(3t)\right)$ 1	
where $4\cos(3t) + 4\sin 3t = x-1$	
$\therefore \ddot{x} = -9(x-1)$ 1 where $n=3, x_0=1$	
ii) $T = \frac{2\pi}{n} = \frac{2\pi}{3}$ 1	1 - correct period
	1 - correct amplitude
$x = 1 + 4\left[\cos(3t) + 4\sin(3t)\right]$	
$x = 1 + 4\left[\sqrt{2}\cos(3t-\alpha)\right]$ $\alpha = \frac{\pi}{4}$	
$\therefore$ amplitude $A = 4\sqrt{2}$ 1	
iii) $d_{MAX} = \lvert x_{MAX}\rvert = 1 + 4\sqrt{2}$ 1	1 - correct answer
iv) $\ddot{x}$ is max. at extremities	1 - correct answer
$\therefore$ at $x = 1+4\sqrt{2}$ or $x = 1-4\sqrt{2}$	(accept $\pm 36\sqrt{2}$)
$\ddot{x} = -9(1+4\sqrt{2}-1) = -36\sqrt{2}$	
or $\ddot{x} = -9(1-4\sqrt{2}-1) = +36\sqrt{2}$	
$\therefore \ddot{x}_{MAX} = 36\sqrt{2}$ 1	

	Marks
v) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9(x-1)$	2 - correct answer
$\frac{1}{2}v^2 = -9\int(x-1)\,dx$	1 - finds expression for v^2
$\frac{1}{2}v^2 = -\frac{9x^2}{2} + 9x + c$	1 - finds equivalent time when $x=2$
when $x = 1+4\sqrt{2}$ $v=0$	
$\therefore v^2 = -9x^2 + 18x + 2c$	
$\therefore 0 = -9(1+4\sqrt{2})^2 + 18(1+4\sqrt{2}) + 2c$	
$2c = 279$	
$\therefore \boxed{v^2 = -9x^2 + 18x + 279}$	
when $x=2$ $\therefore v^2 = -9(2)^2 + 18(2) + 279$	
$\therefore v^2 = 279$	
$\therefore$ speed $= \lvert v\rvert = \sqrt{279} = 3\sqrt{31}$	
or speed $= 16.703$ cm/s	
(OR) when $x=2$	
$2 = 1 + 4\cos(3t) + 4\sin(3t)$	
$2 = 1 + 4\sqrt{2}\cos\left(3t - \frac{\pi}{4}\right)$	
$\frac{1}{4\sqrt{2}} = \cos\left(3t - \frac{\pi}{4}\right)$	
$3t - \frac{\pi}{4} = 1.393\ldots$ (sec)	
$t = 0.72616\ldots$	
$\therefore \lvert\dot{x}\rvert = \lvert -12(\sin 3t - \cos 3t)\rvert$	
$\therefore$ speed $= 16.703$ cm/s	

Marks

c)



$V = \frac{dx}{dt} = 5m/s$

i) $\tan\theta = \frac{x}{100}$ $\therefore$ $x = 100\tan\theta$

$\frac{dx}{d\theta} = 100\sec^2\theta$

OR $\tan\theta = \frac{x}{100}$

$\theta = \tan^{-1}\frac{x}{100}$

$\frac{d\theta}{dx} = \frac{1}{100} \times \frac{1}{1+\left(\frac{x}{100}\right)^2}$ (1) OR $\frac{100}{100^2+x^2}$

$\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$

$= \frac{1}{100} \times \frac{1}{1+\left(\frac{x}{100}\right)^2} \times 5$ (1)

$= \frac{5}{100 \times \frac{100^2+x^2}{100^2}} = \frac{500}{100^2+x^2}$ (1)

ii) when $\theta = \frac{\pi}{4}$ $\therefore$ $x = 100 \times \tan\frac{\pi}{4}$

$\therefore x = 100$

$\therefore \frac{d\theta}{dt} = \frac{500}{100^2+100^2} = \frac{1}{40}$ rad/sec.

Marks:

3 - correct solus

2 - finds correct expression for $\frac{d\theta}{dx}$ or $\frac{dx}{d\theta}$ and attempts to find $\frac{d\theta}{dt}$.

1 - finds another rate relevant to the problem

1 - correct answer with correct units

You may ask for extra writing paper if you need more space to answer question 3