Name: ____________________

Teacher: ____________________

Class: ____________________

FORT STREET HIGH SCHOOL

2006

HIGHER SCHOOL CERTIFICATE COURSE
ASSESSMENT TASK 3

Mathematics Extension 1

TIME ALLOWED: 1 HOUR

Outcomes Assessed	Questions	Marks
Manipulates algebraic expressions and calculus involving logarithmic and exponential functions..	1	
Determines integrals by reduction to a standard form through a given substitution	2	
Uses the relationship between functions, inverse functions and their derivatives.	3	
Evaluates mathematical solutions to problems and communicates them in appropriate form.	All	

Question	1	2	3	Total	%
Marks	/17	/17	/19	/53	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started on a new page

Question 1 (17 marks)

a) Show that $\dfrac{e^{x}}{e^{x}+e^{-x}}=\dfrac{e^{2x}}{e^{2x}+1}$ [2]

b) The curves $y=\dfrac{2e}{x}$ and $y=\log_e x^2$ intersect at $P(e,2)$. Show that the acute angle between these curves is given by $\tan\theta=\dfrac{4e}{e^2-4}$. [4]

c)

i) Show that $\dfrac{x+2}{x+3}=1-\dfrac{1}{x+3}$ [2]

ii) Hence find the value of the definite integral $\displaystyle\int_{-2}^{0}\frac{x+2}{x+3}\,dx$ [2]

d) Prove that $e^{\ln a}=a$ and hence determine the value of $\displaystyle\int_{-\ln 2}^{\ln 2}e^{x}dx$ [4]

e) Find the second derivative of e^{5x^2}. Hence show that the curve $y=e^{5x^2}$ is concave up for all real values of x. [3]

Question 2 (17 marks)

a) Find $\int \frac{t}{\sqrt{1+t}} dt$ using $u = 1+t$ [3]

b) Find $\int \cos^2 4x \quad dx$ [3]

c) Find $\int_0^1 x(5x^2 - 4) \quad dx$ using the substitution $u = 5x^2 - 4$ [3]

d) Evaluate $\int_0^{\frac{\pi}{3}} \tan x \quad dx$ using the substitution $u = \cos x$. [4]

e) Use the substitution $u = x^4 + 4x + 1$ to evaluate to 2 decimal places.

$$\int_0^1 (x^3 + 1) \sqrt[4]{x^4 + 4x + 1} \quad dx$$ [4]

Question 3 (19 marks)

a)

i) Sketch the graph of $x = \sin y$ and on it clearly mark the portion taken as $y = \sin^{-1} x$. [1]

ii) State the domain and range of the function $y = \sin^{-1} 2x$. [2]

b) What is the value of $\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(-1)$. [2]

c) Find the derivative of

i) $\sin^{-1}(\cos x)$ [2]

ii) $\tan^{-1}(x) + \tan^{-1}\left(\dfrac{1}{x}\right)$ [3]

iii) $x \log_e\left(\cos^{-1} x\right)$ [3]

d) Evaluate these integrals

i) $\displaystyle\int \frac{1}{5\sqrt{16-x^2}}\,dx$ [1]

ii) $\displaystyle\int \frac{dx}{x^2+2x+10}$ [2]

iii) $\displaystyle\int_{-3}^{0} \frac{dt}{t^2+9}$ [3]

Question 1 (17 marks)

a) $\frac{e^x}{e^x+e^{-x}} = \frac{e^x}{e^x+e^{-x}} \times \frac{e^x}{e^x}$ ✓

$= \frac{e^{2x}}{e^{2x}+1}$ ✓

Mostly well executed

b) $y = \frac{2e}{x}$ and $y = \log_e x^2$ intersect at $P(e, 2)$

$= 2e\,x^{-1}$

$y = 2\log_e x$

$y' = \frac{2}{x}$ ✓ for both derivatives

$y' = -2e x^{-2}$

at $x = e$, $y' = \frac{2}{e}$

at $x = e$, $y' = \frac{-2e}{e^2}$

$= \frac{-2}{e}$ ✓ for both gradients

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$= \left|\frac{-\frac{2}{e}-\frac{2}{e}}{1+-\frac{2}{e}\times\frac{2}{e}}\right|$ ✓

$= \left|-\frac{4}{e} \div \left(1-\frac{4}{e^2}\right)\right|$

$= \left|\frac{-4}{e} \div \frac{e^2-4}{e^2}\right|$

$= \left|-\frac{4}{e} \times \frac{e^2}{e^2-4}\right|$

$= \left|\frac{-4e}{e^2-4}\right|$ ✓

$= \frac{4e}{e^2-4}$.

Some students did not sub. $x=e$ at this stage which resulted in harder manipulation. Some used the quotient rule for $y = \frac{2e}{x}$ ineffectively

−1 mark for not using absolute value

$\frac{}{x+3} - \frac{}{x+3}$ ✓

$= \frac{x+3}{x+3} - \frac{1}{x+3}$ ✓

$= 1 - \frac{1}{x+3}$.

ii) $\int_{-2}^{0} \frac{x+2}{x+3}\,dx = \int_{-2}^{0}\left(1-\frac{1}{x+3}\right)dx$ ✓

$= \left[x - \ln(x+3)\right]_{-2}^{0}$

$= 0 - \ln 3 - (-2) - \ln 1$ ✓

$= 2 - \ln 3$.

mostly well done

some did not use $\ln 1 = 0$

d) $y = e^{\ln a}$

$\ln y = \ln e^{\ln a}$

$\ln y = \ln a \ln e$ ✓

$\ln y = \ln a$

$y = a$

$\therefore e^{\ln a} = a$. ✓

$\int_{-\ln 2}^{\ln 2} e^x\,dx = \left[e^x\right]_{-\ln 2}^{\ln 2}$ ✓

$= e^{\ln 2} - e^{-\ln 2}$

$= 2 - e^{\ln 2^{-1}}$

$= 2 - \frac{1}{2}$ ✓

$= 1\frac{1}{2}$

Very poorly done by many students either not showing enough detail or working on both sides at same time

Some did not realize $e^{\ln 2^{-1}} = e^{\ln\frac{1}{2}} = \frac{1}{2}$

e) $y = e^{5x^2}$

$y' = 10x\, e^{5x^2}$ ✓

$y'' = 10x \times 10x\, e^{5x^2} + e^{5x^2} \times 10$

$= 10e^{5x^2}(10x^2+1)$ ✓

$y'' > 0$ for all values of x so the curve is always concave up. ✓

Many did not find y'' accurately. Some did not conclude appropriately

Question 2 (17 marks)

a) $\int \frac{t}{\sqrt{1+t}}\,dt$ $u = 1+t$, $t = u-1$, $du = dt$ ✓

$\int \frac{u-1}{\sqrt{u}}\,du$ ✓

$\int (u^{\frac{1}{2}} - u^{-\frac{1}{2}})\,du$

$\frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}} + c$ ✓

$\frac{2\sqrt{(1+t)^3}}{3} - 2\sqrt{1+t} + c$

Mostly well done. Note $\frac{1}{\sqrt{u}} = u^{-\frac{1}{2}}$. Some people left in u form

b) $\int \cos^2 4x\,dx = \frac{1}{2}\int \cos 8x + 1\,dx$ ✓

$= \frac{1}{2}\left(\sin\frac{8x}{8} + x\right) + c$ ✓

$= \frac{\sin 8x}{16} + \frac{x}{2} + c$. ✓

Well done. Some mistakes with substitution $\cos^2 4x = \frac{1}{2}(1+\sin 8$

c) $\int_0^1 x(5x^2-4)\,dx$ $u = 5x^2-4$, $du = 10x\,dx$. If $x=0$, $u=-4$; $x=1$, $u=1$ ✓

$\frac{1}{10}\int_{-4}^{1} u\,du$ ✓

$\left.\frac{u^2}{20}\right]_{-4}^{1}$

$\frac{1}{20} - \frac{16}{20}$

$-\frac{3}{4}$ ✓

Well done

d) $\int_0^{\frac{\pi}{3}} \tan x\,dx$ $u = \cos x$, $du = -\sin x\,dx$. If $x=0$, $u=1$; $x=\frac{\pi}{3}$, $u=\frac{1}{2}$ ✓

$\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x}\,dx$ ✓

$\int_1^{\frac{1}{2}} -\frac{1}{u}\,du$

$\left.-\ln u\right]_1^{\frac{1}{2}}$ ✓

$-\ln\frac{1}{2} - (-\ln 1)$

$\ln 1 - \ln\frac{1}{2}$

$0 + \ln 2$

$\ln 2$ ✓

Some mistakes with $-\int \frac{1}{u}\,du = -\ln u$

e) $\int_0^1 (x^3+1)\sqrt[4]{x^4+4x+1}\,dx$ $u = x^4+4x+1$, $du = 4x^3+4 = 4(x^3+1)$ ✓. If $x=0$, $u=1$; $x=1$, $u=6$

$\frac{1}{4}\int_1^6 u^{\frac{1}{4}}\,du$

$\left.\frac{4u^{\frac{5}{4}}}{20}\right]_1^6$ ✓

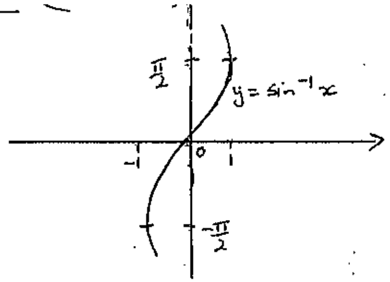
$\frac{1}{5}(6^{\frac{5}{4}} - 1^{\frac{5}{4}})$ ✓

1.678101496

1.68 (to 2 d.p) ✓

Well done overall

a) i)



ii) Domain $-1 \le 2x \le 1$

$\{x : -\frac{1}{2} \le x \le \frac{1}{2}\}$

Range $\{-\frac{\pi}{2} \le y \le \frac{\pi}{2}\}$

b) $\tan^{-1}(\sqrt{3}) - \tan^{-1}(-1)$

$x = \tan^{-1}\sqrt{3}$ $\quad$ $y = \tan^{-1}(-1)$

$\tan x = \sqrt{3}$ $\quad$ $\tan y = -1$

$x = \frac{\pi}{3}$ $\quad$ $y = -\frac{\pi}{4}$

$\tan^{-1}(\sqrt{3}) - \tan^{-1}(-1) = \frac{\pi}{3} - -\frac{\pi}{4}$

$= \frac{7\pi}{12}$

c) i) $\frac{d}{dx} \sin^{-1}(\cos x) = \frac{1}{\sqrt{1-\cos^2 x}} \times -\sin x$

$= \frac{-\sin x}{\sin x}$

$= -1$

i) Some students graphed $y = \sin x$ by mistake

ii) well done

b) Some students used $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$ which did not lead to a result

c) i) well done

$\frac{d}{dx}(\tan^{-1}(x) + \tan^{-1}(\frac{1}{x}))$

$\frac{1}{1+x^2} + \frac{1}{1+(\frac{1}{x})^2} \times -x^{-2}$

$\frac{1}{1+x^2} - \frac{1}{x^2} \div \left(\frac{x^2+1}{x^2}\right)$

$\frac{1}{1+x^2} - \frac{1}{1+x^2}$

0

iii) $\frac{d}{dx}(x \log_e(\cos^{-1}x))$

$x \times \frac{-1}{\sqrt{1-x^2}} \div \cos^{-1}x + \log_e(\cos^{-1}x) \times 1$

$\frac{-x}{(\sqrt{1-x^2})\cos^{-1}x} + \log_e(\cos^{-1}x)$

d) i) $\int \frac{1}{5\sqrt{16-x^2}}\,dx = \frac{1}{5}\sin^{-1}\frac{x}{4} + c$

ii) $\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{(x+1)^2+9}$

$= \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + c$

iii) $\int_{-3}^{0} \frac{dt}{t^2+9} = \frac{1}{3}\left[\tan^{-1}\frac{t}{3}\right]_{-3}^{0}$

$= \frac{1}{3}\left[\tan^{-1}0 - \tan^{-1}-1\right]$

$= \frac{1}{3}(0 - -\frac{\pi}{4})$

$= \frac{\pi}{12}$

Some students failed to use chain rule to find $\frac{d}{dx}(\frac{1}{x})$

iii) well done

d) i) well done

ii) Many students failed to factor $x^2+2x+10 = (x+1)^2+9$, and hence tried to use logs unsuccessfully.

iii) well done