GOSFORD HIGH SCHOOL

SOLNS

Year 12 HSC

Mathematics Extension 1

Assessment Task #3

Time Allowed: *70 Minutes*

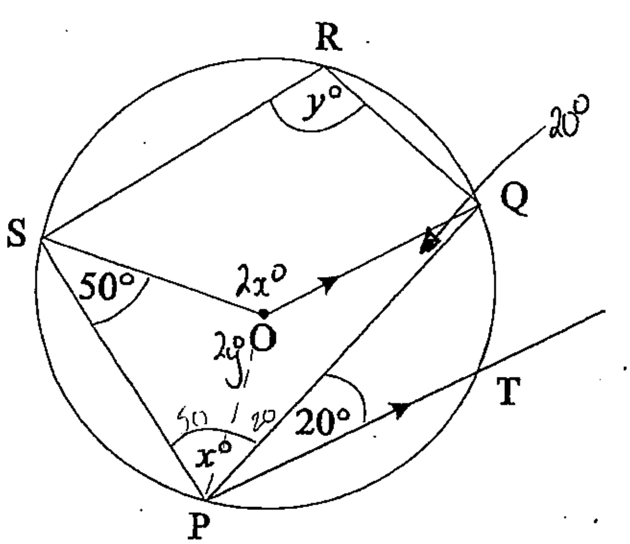
Instructions:

- Attempt all questions.
- Start each question on a new sheet of paper.
- All questions are of equal value.
- Board approved calculators may be used.
- Write using black or blue pen.
- All necessary working should be shown in every question.
- A table of standard integrals is provided at the back of this paper.

Question 1 *(12 marks)* (Start a new sheet of paper.)

Marks

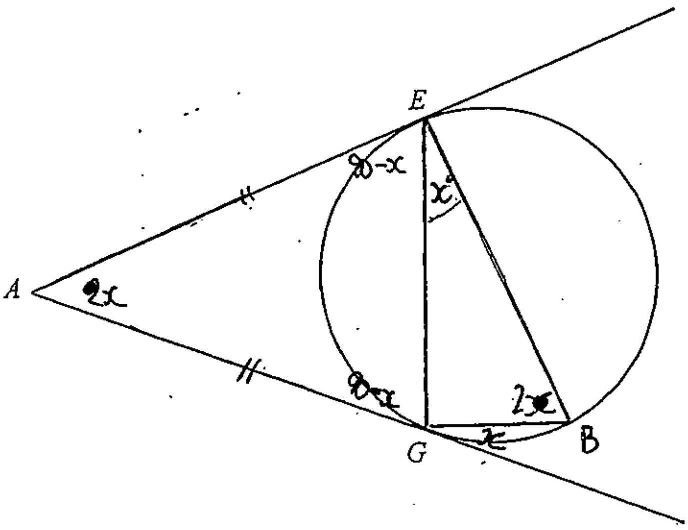
a. On the diagram, O is the centre of the circle and PT | | OQ. P, Q, R and S are points on the circle. Find the values of x and y, giving reasons. 4



$x + 2y = 290$ — 1
$x + y = 180$ — 2
$y = 110^\circ$
$x = 70^\circ$.

Diagram not to scale

b. AE and AG are tangents to a circle. B is a point on the circle such that ∠EBG and ∠EAG are equal and are both double ∠GEB. Let ∠GEB = x°. 4



$2x = 90 - x$ (both tangents)

$x = 30$.

∴ ∠EGB = 90°

Diagram not to scale

i. Find x, giving reasons for your answer.

ii. Hence, prove that EB is a diameter of the circle.

c. In the diagram below, AC is the diameter of circle AECF with centre O and BD is a tangent to the circle at C.

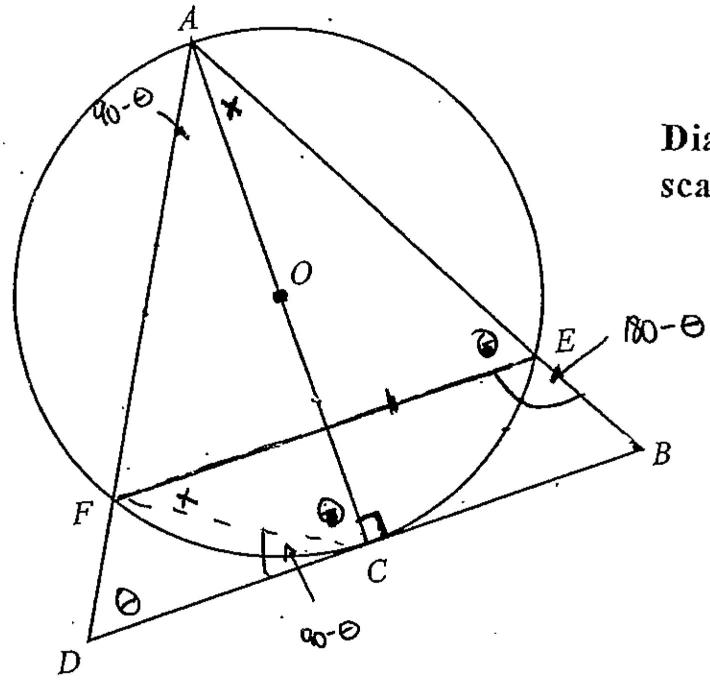


Diagram not to scale

4

i. Neatly copy the diagram on to your answer sheet.

ii. Prove that DFEB is a cyclic quadrilateral.

END OF QUESTION 1

Question 2 *(12 marks)* (Start a new sheet of paper.)

Marks

a. Ryan started a new job for which his starting salary was \$52 000 pa for the first year. As an incentive he receives an increment of \$2500 following each year of service for the first ten years and \$3500 for each year of service after that.

i. Find the amount of salary Ryan will earn in his 18^{th} year of service. 1

ii. What is Ryan's total earnings for the first 18 years of service? 3

b. A farmer borrows \$80 000 to purchase new machinery. The interest is calculated monthly at the rate of 2% per month and is compounded monthly.

The farmer intends to repay the loan over two years making payments of \$M each month.

i. Show that at the end of the first month the farmer owes (in \$) 1

$$80\,000(1.02) - M$$

ii. Deduce that the amount owing at the end of two years is given by (in \$) 2

$$80\,000(1.02)^{24} - M(1 + 1.02^2 + 1.02^3 + ... + 1.02^{23})$$

iii. Find the amount of each monthly repayment. 2

c. Abbie was born on 19^{th} December 2006. On that day her grandparents opened a trust account by depositing \$2000 immediately at birth and \$250 each year thereafter on her birthday..

Abbie is to receive the accumulated value from this account as a 21^{st} Birthday present on the 19^{th} December 2027 immediately after a final payment has been made.

The interest earned on the account is 8% pa compounded every six months.

i. Show that the initial deposit accumulated to \$10 385.57 after 21 years. 1

ii. How much did Abbie's grandparents give her on her 21^{st} Birthday? 2

END OF QUESTION 2

Question 3 *(12 marks)* (Start a new sheet of paper.)

Marks

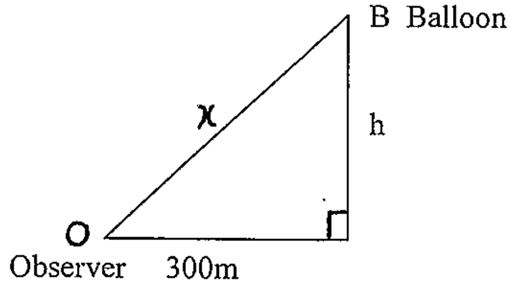
a. A bath tub which holds 240 litres, when full, is drained so that at time t seconds the volume of water V, in litres, is given by:

$$V = 240\left(1 - \frac{t}{60}\right)^2 \text{ for } 0 \le t \le 60$$

i. After how many seconds was the bathtub one-quarter full? 2

ii. At what rate was the water draining out when the bathtub was one-quarter full? 2

b. A weather balloon is rising vertically at 5m/s. An observer is standing on the ground 300m from the point where the balloon was released, as indicated in the diagram. Let h metres be the vertical height of the balloon above the ground and x metres be the distance between the observer and the balloon at any time t secs.



i. Show that $x = \sqrt{h^2 + 90\ 000}$ 1

ii. At what rate is the distance between the observer and the balloon changing when the balloon is 400m high? 2

c. N is the number of kangaroos in a certain population at time t years.

The population size N satisfies the equation

$$\frac{dN}{dt} = -k(N - 500) \quad \textit{(where k is a constant)}$$

i. Verify that $N = 500 + Ae^{-kt}$ *(where A is a constant)* is a solution of the equation. 1

ii. If initially there are 3500 kangaroos and after 3 years there are 3300 find the values of A and k. 2

iii. Find, to the nearest year, when the number of kangaroos begins to fall below 2300. 2

END OF QUESTION 3

Question 4 *(12 marks)* (Start a new sheet of paper.)

Marks

a. The velocity V m/s of a particle moving in a straight line is given by

$$V = \sqrt{4x^5 - 20x^3 + 40x}$$

i. Find an expression for acceleration in terms of x. 1

ii. Hence, find the maximum speed of the particle. 2

b. A particle moves in a straight line from a fixed point, 0, along the x axis. Its velocity V m/s can be found using:

$$V^2 = x^2 - 9$$

Initially the particle is located 5m to the right of 0 and is moving towards the origin.

i. Find the acceleration of the particle in terms of x. 1

ii. Show that the particle does not pass through the origin. 1

iii. Find the set of possible values of x. 2

iv. Describe the motion of the particle. 1

c. The acceleration of a particle moving in a straight line is given by:

$$a = -16e^{-4x}$$

Initially the particle is at the origin with velocity $2\sqrt{2}$ m/s.

i. Prove that $v = 2\sqrt{2}e^{-2x}$ 2

ii. Hence, show that $x = \frac{1}{2}\ln(1 + 4\sqrt{2}t)$. 2

END OF ASSESSMENT TASK

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

GHS – MATHEMATICS EXTENSION 1

ASSESSMENT TASK #3 SOLUTIONS

QUESTION 1.

a. $O\hat{Q}P = 20^\circ$ (alternate ∠'s in ∥ are =)

reflex $S\hat{O}Q = 2x^\circ$ (∠ at the centre of ⊙ is twice ∠ at circumference standing on same arc)

∴ $S\hat{O}Q = 360^\circ - 2x^\circ$ (∠'s at revolution = 360°)

∴ $50^\circ + x^\circ + 20^\circ + (360^\circ - 2x^\circ) = 360^\circ$ (∠ sum of quadrilateral = 360°)

∴ $430 - x = 360$

$70 = x$

$y = 110$ (opposite ∠'s in cyclic quadrilateral are supplementary.)

b. (i) $E\hat{B}G = E\hat{A}G = 2x^\circ$ (given)

∴ Since △AEG is isosceles. (tangents drawn from an external point are equal)

$A\hat{E}G = 90^\circ - x^\circ$ (equal base ∠'s in isosceles △, ∠ sum of △ = 180°)

∴ $E\hat{B}G = 90^\circ - x^\circ$ (∠ between chord and tangent is = ∠ in the alternate segment)

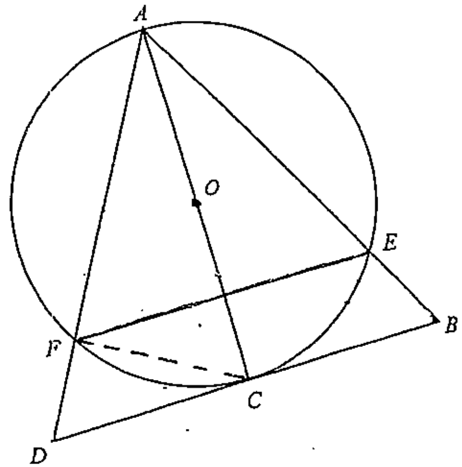
∴ $2x = 90 - x$

$3x = 90$

$x = 30$

(ii) $E\hat{G}B = 90^\circ$ (∠ sum of △ = 180°)

c. (i)



(ii) Construct a line interval FC.

let $A\hat{C}F = x^\circ$.

∴ $A\hat{E}F = x^\circ$ (∠'s at the circumference standing on the same arc are equal)

∴ $B\hat{E}F = 180^\circ - x^\circ$ (straight ∠)

Now $A\hat{F}C = 90^\circ$ (∠ in semicircle = 90°, AC diameter given)

∴ $C\hat{F}D = 90^\circ$ (straight ∠)

and $F\hat{C}D = 90^\circ - x^\circ$ (∠ between tangent and a radius (AC is diameter) = 90°)

∴ $F\hat{D}C = x^\circ$ (∠ sum of △FCD = 180°)

Since $F\hat{D}C$ and $B\hat{E}F$ are supplementary DFEB is a cyclic quadrilateral (opposite ∠'s in cyclic quadrilateral are supplementary)

QUESTION 2

a.

(i) note that in first ten years there are 9 increments

∴ $T_{18} = 52000 + 9\times2500 + 8\times3500$

$= 102500$

∴ In his 18th year Ryan will receive a salary of $102 500

(ii) Ryan's total earnings ⇒ $A_1 + A_2$

$(T_1 - T_{10})\ A_1 = 52000 + 52000 + 1\times2500 + \dots + 52000 + 9\times2500$

$(T_{11} - T_{18})\ A_2 = 74500 + 1\times3500 + \dots + 74500 + 8\times3500$

(using $S_n = \frac{n}{2}(a+L)$)

$A_1 = \frac{10}{2}(52000 + 74500) = 632500$

$A_2 = \frac{8}{2}(78000 + 102500) = 722000$

Total $= 632500 + 722000$

$= \$1\,354\,500$.

b. (i) using $A = P(1+\frac{r}{100})^n$ – Repayment

$A_1 = 80000(1+0.02)^1 - M$

$= 80000(1.02) - M$ as req.

(ii) $A_2 = A_1(1.02) - M$

$= [80000(1.02) - M](1.02) - M$

$= 80000(1.02)^2 - (1.02)M - M$

$= 80000(1.02)^2 - M(1+1.02)$

∴ $A_3 = 80000(1.02)^3 - M(1+1.02+1.02^2)$

∴ $A_{24} = 80000(1.02)^{24} - M(1+1.02+\dots+1.02^{23})$, as required

(iii) Now $A_{24} = 0$

∴ $M = \dfrac{80000(1.02)^{24}}{1+1.02+\dots+1.02^{23}}$

Using $S_n = \frac{a(r^n-1)}{r-1}$ for series

$S_{24} = \dfrac{1(1.02^{24}-1)}{1.02-1} = 30.42186\dots$ (calc)

∴ $M = 4229.68778\dots$ (calc)

∴ Each repayment is $4229.69

c. (i) Using $A = P(1+\frac{r}{100})^n$

$A_1 = 2000(1.04)^{42} = \$10\,385.57$ as req.

(ii) $A_2 = 250(1.04)^{40}$

$A_3 = 250(1.04)^{38}$

$A_4 = 250(1.04)^{36}$ and so on...

.....

$A_{20} = 250(1.04)^2$

$A_{21} = 250$

∴ $A_{21} + A_{20} + A_{19} + \dots + A_2$ forms a GP.

$S_n = \dfrac{a(r^n-1)}{r-1}$

$= \dfrac{250\times[(1.04)^{20}-1]}{(1.04)^2-1}$

$= \$3649.27$

Total $= \$10\,385.57 + \$3649.27$

QUESTION 3.

a. $V = 240\left(1-\frac{t}{60}\right)^2$

(i) $60 = 240\left(1-\frac{t}{60}\right)^2$

$\frac{1}{4} = \left(1-\frac{t}{60}\right)^2$

$\pm\frac{1}{2} = 1-\frac{t}{60}$

$\frac{t}{60} = 1 \pm \frac{1}{2}$

$\frac{t}{60} = \frac{3}{2}, \frac{1}{2}$

$t = \cancel{90}, 30$ outside time range

$\therefore$ bathtub is one quarter full after 30 seconds

(ii) $\frac{dv}{dt} = 480\left(1-\frac{t}{60}\right) \times -\frac{1}{60}$

$= -8\left(1-\frac{t}{60}\right)$

(when $t=30$)

$\frac{dv}{dt} = -8\left(1-\frac{1}{2}\right)$

$= -4$

$\therefore$ the water is draining out at a rate of 4 litres/sec.

b. (i) Using Pythagoras' Theorem

$x^2 = h^2 + 300^2$

$= h^2 + 90\,000$

$\therefore x = \sqrt{h^2+90\,000}$ as required

(ii) to find $\frac{dx}{dt} = \frac{dx}{dh} \times \frac{dh}{dt}$

Now $x = (h^2+90000)^{1/2}$

$\frac{dx}{dh} = \frac{1}{2}(h^2+90000)^{-1/2} \times 2h$

$= \frac{h}{\sqrt{h^2+90000}}$

(when $h=400$)

$\frac{dx}{dh} = \frac{400}{\sqrt{400^2+90000}}$

$= \frac{400}{500}$

$= \frac{4}{5}$

$\therefore \frac{dx}{dt} = \frac{4}{5} \times 5$

$= 4$

the distance is increasing at 4 m/s.

c. (i) If $N = 500 + Ae^{-kt}$ — ①

$\frac{dN}{dt} = -kAe^{-kt}$

$= -k(N-500)$ as req.

Since $N-5 = Ae^{-kt}$ from ①

(ii) when $t=0$ $N = 3500$

$\therefore 3500 = 500 + A$

$\therefore A = 3000$

(when $t=3$) $3300 = 500 + 3000e^{-3k}$

$2800 = 3000e^{-3k}$

$\frac{14}{15} = e^{-3k}$

$\ln\left(\frac{14}{15}\right) = -3k$

$-\frac{1}{3}\ln\left(\frac{14}{15}\right) = k$

$\therefore k = 0.022997623...$ (calc)

$\therefore k = 0.023$ (3 d.p)

(iii) $2300 = 500 + 3000e^{-kt}$

$1800 = 3000e^{-kt}$

$\frac{3}{5} = e^{-kt}$

$\ln\left(\frac{3}{5}\right) = -kt$

$-\frac{1}{k}\ln\left(\frac{3}{5}\right) = t$

$\therefore t = 22.2121045...$ (calc)

$\therefore$ after 22 years

QUESTION 4

a. (i) $v = \sqrt{4x^5 - 20x^3 + 40x}$

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}(4x^5 - 20x^3 + 40x)\right)$

$= \frac{d}{dx}\left(2x^5 - 10x^3 + 20x\right)$

$= 10x^4 - 30x^2 + 20$

(ii) Max. speed occurs $a=0$

i.e. $10x^4 - 30x^2 + 20 = 0$

$x^4 - 3x^2 + 2 = 0$

$(x^2-2)(x^2-1) = 0$

$\therefore x = \pm\sqrt{2}, \pm 1$

Max. speed occurs when $|v|$ is greatest.

(motion not possible if $x = -\sqrt{2}, -1$)

$\therefore x = 1 \Rightarrow v = 2\sqrt{6}$

$x = \sqrt{2} \Rightarrow v = 4\sqrt[4]{2}$

Since $2\sqrt{6} > 4\sqrt[4]{2}$, Max speed is $2\sqrt{6}$.

b. (i) $\frac{1}{2}v^2 = \frac{1}{2}x^2 - \frac{9}{2}$ $\left(a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)\right)$

$\therefore a = x$

(ii) ($x=0$) $v^2 = 0-9$ $\therefore v^2 = -9$ which can't happen. $\therefore$ the particle does not pass through the o

(iii) For motion to exist $v^2 \geq 0$

$\therefore x^2 - 9 \geq 0$

$x \leq -3$ or $x \geq 3$

but when $t=0$, $x=5$ $\therefore x \leq -3$ is not possible and so $x \geq 3$ are the possible values.

(iv) The particle starts 5m to the right of and is moving towards 0. It continues to mo to the left until it reaches $x=3$ where it stops ($v=0$). It then moves to the right and speeds up ($a>0$) continuing to move to the right.

c. (i) $a = -16e^{-4x}$

$\frac{1}{2}v^2 = -16\int e^{-4x}\,dx$

$\therefore \frac{1}{2}v^2 = 4e^{-4x} + C_1$

($x=0, v=2\sqrt{2}$) $4 = 4 + C_1$ $\therefore C_1 = 0$

$v^2 = 8e^{-4x}$ $\therefore v = \pm 2\sqrt{2}e^{-2x}$

(Since $x=0, v=2\sqrt{2}$) $\therefore v = 2\sqrt{2}e^{-2x}$ as re

(ii) $\frac{dx}{dt} = 2\sqrt{2}e^{-2x}$

$\frac{dt}{dx} = \frac{e^{2x}}{2\sqrt{2}}$

$t = \frac{e^{2x}}{4\sqrt{2}} + C_2$

($t=0, x=0$) $\therefore C_2 = -\frac{1}{4\sqrt{2}}$

$t = \frac{e^{2x}}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}$

$4\sqrt{2}t = e^{2x} - 1$

$4\sqrt{2}t + 1 = e^{2x}$

$\ln(4\sqrt{2}t+1) = 2$

$\therefore x = \frac{1}{2}\ln(1+4\sqrt{}$

as req.