GOSFORD HIGH SCHOOL

2009

ASSESSMENT TASK 3

MATHEMATICS EXTENSION 1

Time allowed: *70 minutes (plus 5 minutes reading time)*

Name: ______________________________

General Instructions:

- Write using a black or blue pen.
- Show all necessary working.
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.

TOTAL MARKS: 50

- Attempt all Sections 1 – 4
- Sections are **NOT** of equal value.
- Start each **Section** on a new sheet of paper.

MARKERS USE ONLY		
Section 1:		**Total**
I. Integration		
II. Series		/14
Section 2:		
I. Parametric Representation		
II. Mathematical Induction		/12
Section 3: Inverse Functions		/10
Section 4: Inverse Trigonometric Functions		/14
TOTAL		/50

SECTION 1 (14 Marks) Use a new sheet of paper.

Marks

I. **INTEGRATION**

a. Find $\int \cos^2 3x \, dx$ 2

b. Evaluate, in exact form: 3

$$\int_0^1 x(1-2x)^3 dx$$

Using the substitution $u = 1 - 2x$

c. By using $u = 1 + e^x$, show that 3

$$\int_0^1 \frac{e^{3x}}{1+e^x} \, dx = \frac{(e-1)^2}{2} + \ln\left(\frac{e+1}{2}\right)$$

Section 1 continued

II. SERIES

a. If a, b and c are consecutive terms in an arithmetic series, show that e^a, e^b *and* e^c are consecutive terms in a geometric series. 1

b. Find the sum of the first 15 terms of the series: 2

$$\log_{10} 3 + \log_{10} 27 + \log_{10} 243 + ...$$

(Answer in exact form)

c. For the infinite geometric series:

$$2 + \frac{4}{x+5} + \frac{8}{(x+5)^2} + ...$$

i. Find the limiting sum when $x = 5$ 1

ii. For what values of x will the above series have a limiting sum? 2

SECTION 2 (12 Marks) Use a new sheet of paper.

Marks

I. PARAMETRIC REPRESENTATION

a. Given the parametric equations

$x = 8t$ and $y = 4t^2$ 1

Eliminate t to find the Cartesian equation of the parabola

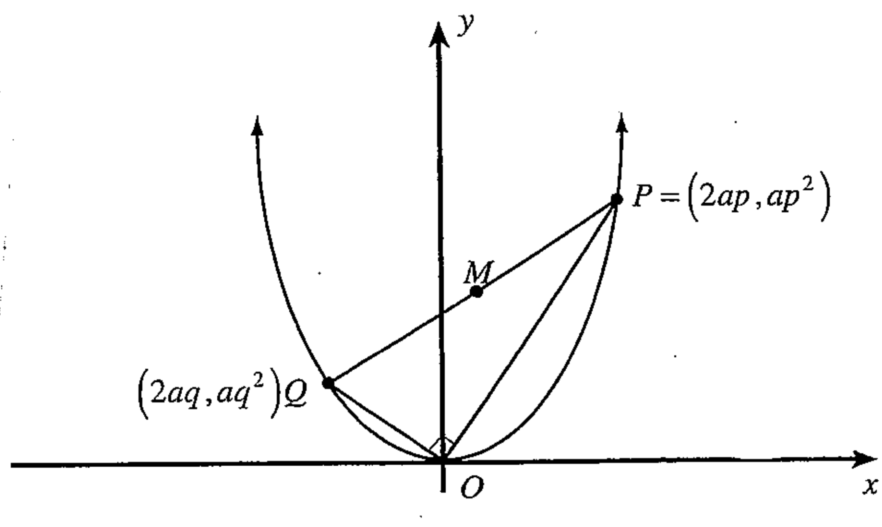
b. Given the parabola $x^2 = 4y$

i. Show that the equation of the tangent at the point $P(2t, t^2)$ is given by $y = tx - t^2$ 2

ii. Find the equation of the two tangents to the parabola that could be drawn from the point $(1, -6)$ 2

iii. Find the co-ordinates of the two points of contact of these tangents. 1

c. $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$. PQ subtends a right angle at the vertex O of the parabola.



i. Prove that pq = -4 1

ii. Find the equation of the locus of the midpoint M of the chord PQ. 2

Section 2 Continued

Marks

II. MATHEMATICAL INDUCTION

a. Show by mathematical induction that for all positive integers $n \geq 1$

$$\cos(x + n\pi) = (-1)^n \cos x$$ 3

SECTION 3 (10 Marks) Use a new sheet of paper.

Marks

INVERSE FUNCTIONS

a. Consider the function $y = x^2 - 2$

i. Sketch y = f(x) showing x and y intercepts. 1

ii. Find the largest positive domain for which $y = f(x)$ has an inverse function $y = f^{-1}(x)$ 1

iii. State the domain of $y = f^{-1}(x)$ 1

iv. Sketch the graph of $y = f^{-1}(x)$ in this domain 1

b. Given the function of $g(x) = 2 + \dfrac{4}{x-3}$ for $x > 3$

i. Sketch the function $y = g(x)$ clearly indicating all intercepts and asymptotes. 2

ii. Find an expression for the inverse function $y = g^{-1}(x)$ in terms of x 2

iii. Find the x co-ordinate, correct to 2 decimal places, of the point of intersection of $y = g(x)$ and $y = g^{-1}(x)$ 2

SECTION 4 (14 Marks) Use a new sheet of paper.

Marks

INVERSE TRIGONOMETRIC FUNCTIONS

a. Given that $f(x) = 3\cos^{-1}\frac{x}{2}$, evaluate *f(1)* in exact form — 1

b. Find $\frac{dy}{dx}$ if $y = (2x+1)^3 \tan^{-1} x$ — 2

c. Find $\int \frac{dx}{\sqrt{1-49x^2}}$ — 2

d. The portion of the curve $y = \frac{1}{\sqrt{4+x^2}}$ between *x=0* and *x*=2 is rotated about the *x* axis. Find the exact volume of the solid of revolution formed. — 2

e. Let *h(x)* = $\sin^{-1}(x+3)$

i. State the domain and range of *h(x)* — 2

ii. Find the equation of the normal to the curve *y=h(x)* at the point where $x = -3$ — 2

iii. Sketch the graph of *y=h(x)* — 1

f. Evaluate $\sin\left[\cos^{-1}(\frac{4}{5}) + \tan^{-1}(-\frac{4}{3})\right]$ — 2

End of Test

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

SECTION I

1. Integration

a. $\int \cos^2 3x \, dx$

$= \frac{1}{2}\int (1+\cos 6x)\,dx$

$= \frac{1}{2}\left(x + \frac{\sin 6x}{6}\right) + c$

b. $\int_0^1 x(1-2x)^3 \, dx$

$u = 1-2x$, $\frac{du}{dx} = -2$ $\therefore dx = -\frac{du}{2}$

Also $x = \frac{1-u}{2}$

$x=1$ $u=-1$; $x=0$ $u=1$

$= \int_1^{-1} \frac{1-u}{2}(u)^3 \cdot -\frac{du}{2}$

$= \frac{1}{4}\int_{-1}^{1} (1-u)u^3 \, du$

$= \frac{1}{4}\int_{-1}^{1} u^3 - u^4 \, du$

$= \frac{1}{4}\left[\frac{u^4}{4} - \frac{u^5}{5}\right]_{-1}^{1}$

$= \frac{1}{4}\left[\left(\frac{1}{4}-\frac{1}{5}\right) - \left(\frac{1}{4}+\frac{1}{5}\right)\right] = -\frac{1}{\ldots}$

c. $\int_0^1 \frac{e^{3x}}{1+e^x}\,dx$

$u = 1+e^x$, $\frac{du}{dx} = e^x$, $\frac{du}{e^x} = dx$

$x=1$ $u=1+e$; $x=0$ $u=2$

$= \int_2^{1+e} \frac{e^{2x}}{u}\,du$

(Now $e^x = u-1$ $\therefore e^{2x} = (u-1)^2$)

$= \int_2^{1+e} \frac{(u-1)^2}{u}\,du$

$= \int_2^{1+e} \frac{u^2-2u+1}{u}\,du$

$= \int_2^{1+e} u - 2 + \frac{1}{u}\,du$

$= \left[\frac{u^2}{2} - 2u + \ln u\right]_2^{1+e}$

$= \left[\left(\frac{(1+e)^2}{2} - 2(1+e) + \ln(1+e)\right) - \left(2 - 4 + \ln 2\right)\right]$

$= \frac{1+2e+e^2-4-4e}{2} + \ln\left(\frac{1+e}{2}\right) + 2$

$= \frac{(e-1)^2}{\ldots} + \ln(1+e)$ as re...

11. Series

a. $b-a = c-b$

$\therefore e^{b-a} = e^{c-b}$

$\frac{e^b}{e^a} = \frac{e^c}{e^b}$

$\therefore e^a, e^b, e^c$ form a G.P.

b. $\log_{10}3 + \log_{10}3^3 + \log_{10}3^5 + \ldots$

$= \log_{10}3 + 3\log_{10}3 + 5\log_{10}3 + \ldots$

$\therefore$ Arithmetic with $a = \log_{10}3$, $d = 2\log_{10}3$

$\therefore S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$S_{15} = \frac{15}{2}\left[2\log_{10}3 + 14(2\log_{10}3)\right]$

$= 225\log_{10}3$.

c. $2 + \frac{4}{x+5} + \frac{8}{(x+5)^2} + \ldots$

(i) when $x = 5$

$2 + \frac{4}{10} + \frac{8}{100} + \ldots$

$S_\infty = \frac{a}{1-r}$

$= \frac{2}{1-\frac{2}{10}}$

$= \frac{5}{2}$

(ii) The series will have a S_∞ when $-1 < r < 1$ or $|r| < 1$

i.e $\left|\frac{2}{x+5}\right| < 1$ CV: $x \neq$

Consider

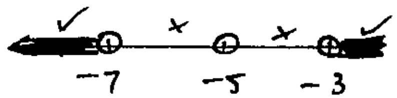
$\left|\frac{2}{x+5}\right| = 1$

i.e $\frac{2}{x+5} = 1$ or $\frac{2}{x+5} = \ldots$

$2 = x+5$ $\qquad$ $2 = -x-5$

$-3 = x$ $\qquad$ $x = -7$

Test



i.e $x < -7$ or $x > -3$

SECTION 2

1. Parametric Representation

a. $x = 8t \quad -(1)$

$y = 4t^2 \quad -(2)$

From (1) $t = \frac{x}{8}$

Sub in (2) $y = \frac{x^2}{16}$

$\therefore x^2 = 16y$

b. (i) $y = \frac{x^2}{4}$

$\therefore \frac{dy}{dx} = \frac{x}{2}$

when $x = 2t$, $m = t$

$\therefore$ eqn of tangent:

$y - t^2 = t(x - 2t)$

$y - t^2 = tx - 2t^2$

$y = tx - t^2$ as req.

(ii) $x = 1, y = -6$

$\therefore -6 = t - t^2$

$t^2 - t - 6 = 0$

$(t-3)(t+2) = 0$

$\therefore t = 3, -2$

$\therefore$ 2 tangents are:

$y = 3x - 9$ and

$y = -2x - 4$

(iii) $t = 3 \quad P = (6, 9)$

$t = -2 \quad P = (-4, 4)$

c. (i) $M_{OP} = \frac{ap^2 - 0}{2ap - 0}$

$= \frac{ap^2}{2ap}$

$= \frac{p}{2}$

Similarly $M_{OQ} = \frac{q}{2}$

If $PO \perp QO$ then

$\frac{p}{2} \times \frac{q}{2} = -1$

$\therefore pq = -4$ as req.

(ii) $M = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$= \left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$

$\therefore x = a(p+q) \quad -(1)$

$y = \frac{a(p^2+q^2)}{2} \quad -(2)$

$(2) \Rightarrow y = \frac{a}{2}\left[(p+q)^2 - 2pq\right]$

Since from c(i) $pq = -4$

and from (1) $p + q = \frac{x}{a}$

$(2) \Rightarrow y = \frac{a}{2}\left[\left(\frac{x}{a}\right)^2 + 8\right]$

$y = \frac{x^2}{2a} + 4a$

$2a(y - 4a) = x^2$

$\therefore x^2 = 2ay - 8a^2$

II. Mathematical Induction

Step 1. To prove true for $n = 1$

LHS $= \cos(x + \pi) = -\cos x$

RHS $= (-1)^1 \cos x = -\cos x$

$\therefore$ LHS = RHS

and statement is true for $n = 1$.

Step 2. Assume it is true for $n = k$

i.e. $\cos(x + k\pi) = (-1)^k \cos x$

Step 3. To prove true for $n = k + 1$

LHS: $\cos(x + (k+1)\pi)$

$= \cos((x + k\pi) + \pi)$

$= \cos(x + k\pi)\cos\pi - \sin(x + k\pi)\sin\pi$

$= (-1)^k \cos x \cdot (-1) - \sin(x + k\pi) \cdot 0$

$= (-1)^{k+1} \cos x$

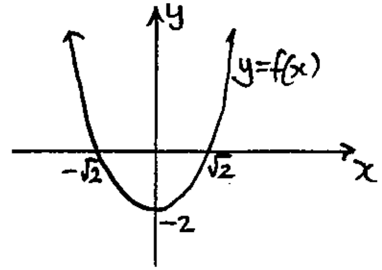
$\therefore$ result holds for $n = k+1$ if true for $n = k$.

$\therefore$ proved by induction

SECTION 3

INVERSE FUNCTIONS

a. (i) $y = x^2 - 2$

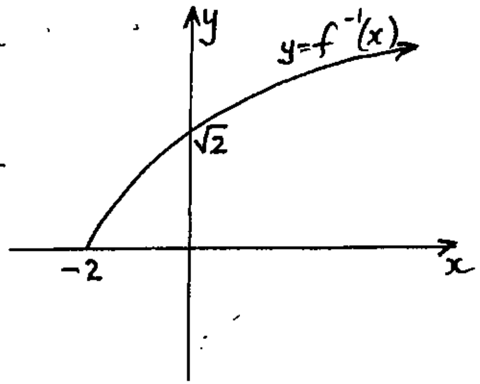


(ii) $x \geq 0$

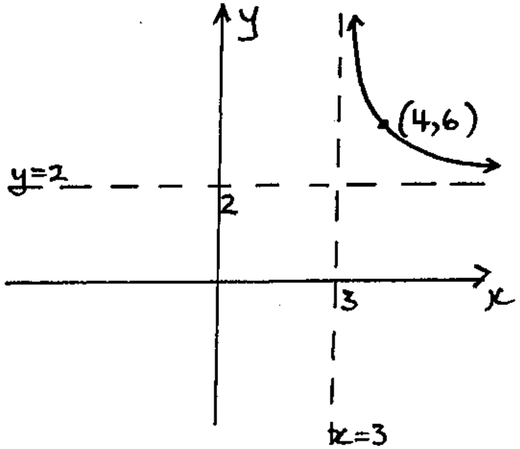
(iii) Domain of $y = f^{-1}(x)$ is the range of $y = f(x)$

$\therefore$ Domain is $x \geq -2$

iv.



b. (i) $g(x) = 2 + \frac{4}{x-3}$



(ii) $y = 2 + \frac{4}{x-3}$

$g^{-1}(x) \Rightarrow \quad x = 2 + \frac{4}{y-3}$

$x - 2 = \frac{4}{y-3}$

$(x-2)(y-3) = 4$

$xy - 3x - 2y + 6 = 4$

$y(x-2) = 3x - 2$

$y = \frac{3x-2}{x-2}$

(iii) To find the point of intersection

solve $y = x \quad -(1)$

$y = \frac{3x-2}{x-2} \quad -(2)$

Sub (1) in (2)

$x = \frac{3x-2}{x-2}$

$x^2 - 2x = 3x - 2$

$x^2 - 5x + 2 = 0$

$x = \frac{+5 \pm \sqrt{25-8}}{2}$

$= \frac{5 \pm \sqrt{17}}{2}$

$= \frac{5+\sqrt{17}}{2}$, ~~$\frac{5-\sqrt{17}}{2}$~~ not in req. domain

$\therefore \quad x = 4.56$ (2d.p)

SECTION 4

Inverse Trig. Functions

a. $f(x) = 3\cos^{-1}\frac{x}{2}$

$f(1) = 3\cos^{-1}\frac{1}{2}$

$= 3 \cdot \frac{\pi}{3}$

$= \pi$

b. $y = (2x+1)^3 \tan^{-1} x$

$\frac{dy}{dx} = \tan^{-1}x \cdot 3(2x+1)^2 \cdot 2 + (2x+1)^3 \cdot \frac{1}{1+x^2}$

$= 6(2x+1)^2 \tan^{-1}x + \frac{(2x+1)^3}{1+x^2}$

c. $\int \frac{dx}{\sqrt{1-49x^2}}$

$= \int \frac{dx}{\sqrt{49\left(\frac{1}{49} - x^2\right)}}$

$= \frac{1}{7}\sin^{-1}7x + C$

d. $V = \pi \int_0^2 \left(\frac{1}{\sqrt{4+x^2}}\right)^2 dx$

$= \pi \int_0^2 \frac{1}{4+x^2}\, dx$

$= \pi \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$

$= \frac{\pi}{2}\left[\tan^{-1}1 - \tan^{-1}0\right]$

$= \frac{\pi}{2}\left[\frac{\pi}{4} - 0\right]$

$= \frac{\pi^2}{8}$ units^3

e. (i) $h(x) = \sin^{-1}(x+3)$

Domain : $-1 \le x+3 \le 1$

$-4 \le x \le -2$

Range : $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(ii) $y = \sin^{-1}(x+3)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x+3)^2}} \times 1$

$$= \frac{1}{\sqrt{1-(x+3)^2}}$$

when $x = -3$

$$\frac{dy}{dx} = 1$$

$\therefore m_{normal} = -1$

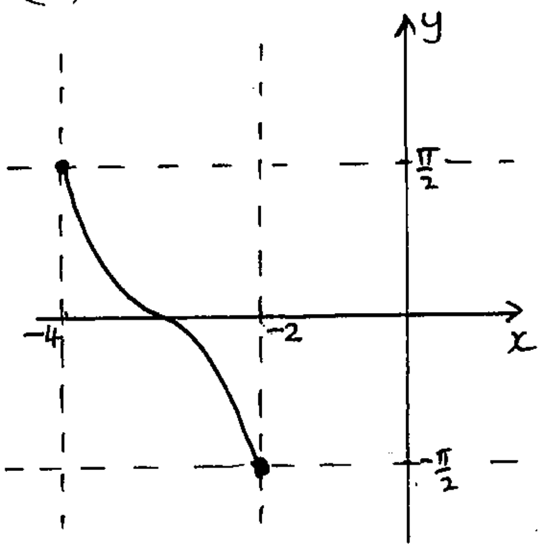
and equation of normal is

$$y - 0 = -1(x+3)$$

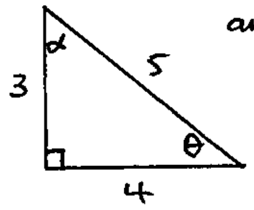
$$y = -x - 3$$

$$x + y + 3 = 0$$

(iii)



(f) Using:



and $\tan^{-1}(-x) = -\tan^{-1}x$

$\cos\theta = \frac{4}{5}$ $\qquad$ $\tan\alpha = \frac{4}{3}$

$\therefore \cos^{-1}\frac{4}{5} = \theta$ $\qquad$ $\tan^{-1}\frac{4}{3} = \alpha$

$\therefore \tan^{-1}\left(-\frac{4}{3}\right) = -\alpha$

$$\therefore \sin\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(-\frac{4}{3}\right)\right]$$

$$= \sin(\theta - \alpha)$$

$$= \sin\theta\cos\alpha - \sin\alpha\cos\theta$$

$$= \frac{3}{5}\cdot\frac{3}{5} - \frac{4}{5}\cdot\frac{4}{5}$$

$$= \frac{9}{25} - \frac{16}{25}$$

$$= -\frac{7}{25}.$$