GOSFORD HIGH SCHOOL

2010 HIGHER SCHOOL CERTIFICATE

MATHEMATICS

EXTENSION 1

ASSESSMENT TASK 3

Time Allowed – 60 minutes + 5 minutes reading time

Students must begin each new question on a new page.

All questions are NOT of equal value.

All questions are to be attempted.

All necessary working should be shown.

Marks may be deducted for careless or badly arranged work.

Board approved calculators may be used.

Questions will be collected separately at the conclusion of the assessment task.

QUESTION 1 (20 marks)

(a) Use the given table of Standard Integrals to find $\int \frac{1}{\sqrt{x^2+1}}dx$ (1)

(b) Find $\int \cos x \cdot \sin^2 x\, dx$ (1)

(c) Find $\int \frac{dx}{x\ln x}$ using the substitution $u = \ln x$ (2)

(d) Evaluate $\int_0^3 \frac{x+2}{\sqrt{x+1}}dx$ using $u^2 = x+1$ (4)

(e) (i) Sketch the curve $y = 1+\sin x$ for $0 \le x \le \frac{3\pi}{2}$ (2)

(ii) The area bounded by this curve and the coordinate axes is rotated about the x axis. Show that the volume of the solid generated is $\frac{\pi}{4}(9\pi+8)$ cubic units. (3)

(f) Find $\int \frac{dx}{\sqrt{x(1-x)}}$ using the substitution $x = u + \frac{1}{2}$ (3)

(g) (i) Write down the general solutions to the equation $\cos 4\theta = -\frac{\sqrt{3}}{2}$ (2)

(ii) Hence, or otherwise find all solutions of $\cos 4\theta = -\frac{\sqrt{3}}{2}$ in the interval $0 \le \theta \le 2\pi$ (2)

QUESTION 2 (21 marks)

(a) $f(x) = 2\sin^{-1}(3x)$,

(i) state the domain and range of $f(x)$ (2)

(ii) find the exact value of $f(\frac{1}{6})$ (1)

(iii) Show that $f'(\frac{1}{6}) = 4\sqrt{3}$ (2)

(b) If $\frac{dy}{dx} = \frac{1}{1+x^2}$ and $y = 0$ when $x = 1$, find y when $x = \sqrt{3}$ (3)

(c) Find $\int \frac{dx}{\sqrt{9-16x^2}}$ (2)

(d) Find the exact value of $\sin\left[2\cos^{-1}(-\frac{5}{7})\right]$ (3)

(e) (i) Sketch the function $f(x) = x^2 - 2x - 3$, clearly labeling all intercepts with the co-ordinate axes. (use the same scale on both axes.) (2)

(ii) State the largest possible domain of $f(x)$, consisting of positive values of x, for which $f(x)$ has an inverse $f^{-1}(x)$ (1)

(iii) Find $f^{-1}(x)$ (3)

(iv) Sketch $y = f^{-1}(x)$ on the same graph as $y = f(x)$ (2)

QUESTION 3 (18 marks)

(a) Given $\frac{dx}{dt} = (3-x)^2$ and $x = 2$ when $t = 0$, find x in terms of t (4)

(b) A spherical bubble is expanding so that its volume is increasing at $10\,\text{cm}^3/\text{s}$
Find the rate of increase of the surface area when the radius is 3cm. (3)

(c) Newton's law of cooling states that the rate at which a body loses heat is proportional to the difference between the temperature(T) of the body and that of the temperature (R) of the surrounding medium.

That is $\frac{dT}{dt} = -k(T-R)$.

(i) Show that $T = R + Ce^{-kt}$, where C and k are constants satisfies this differential equation. (1)

(ii) A body whose temperature is $200°C$ is immersed in a liquid kept at a constant temperature of $50°C$. In one minute, the temperature of the immersed body falls to $150°C$.

Find the time taken for body to reach $90°C$. (4)

(d) An object has velocity $v\,\text{ms}^{-1}$ and acceleration $\frac{d^2x}{dt^2}\,\text{ms}^{-2}$ at position x m from the origin

(i) Show that $\frac{d}{dx}\left[\frac{v^2}{2}\right] = \frac{d^2x}{dt^2}$ (1)

(ii) The acceleration in ms^{-2} of an object is given by $\frac{d^2x}{dt^2} = 2x^3 + 4x$

If the object is initially 2m to the right of the origin travelling with velocity 6ms^{-1}, find an expression for v^2 in terms of x. (3)

(iii) State the minimum speed of the object, justifying your answer. (2)

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Q3)

a) $\frac{dx}{dt} = (3-x)^2$

$\frac{dt}{dx} = \frac{1}{(3-x)^2}$

$t = \frac{1}{3-x} + c$

$x=2,\ t=0$

$0 = 1 + c$

$\therefore c = -1$

$t = \frac{1}{3-x} - 1$

$\frac{1}{3-x} = t+1$ (4)

$3-x = \frac{1}{t+1}$

$x = 3 - \frac{1}{t+1}$

$x = \frac{3t+2}{t+1}.$

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b) $\frac{dV}{dt} = 10$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$10 = 4\pi r^2 \times \frac{dr}{dt}$ (3)

$\frac{dr}{dt} = \frac{5}{2\pi r^2}$

$A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 8\pi r \times \frac{5}{2\pi r^2}$

$= \frac{20}{r}$

when $r = 3$

$\frac{dA}{dt} = \frac{20}{3}$ cm²/s.

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c) i) $T = R + Ce^{-kt}$

$\frac{dT}{dt} = -Cke^{-kt}$ (1)

$= -k(Ce^{-kt})$

$= -k(T-R)$

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ii) $T = R + Ce^{-kt}$

$T = 50 + Ce^{-kt}$

$t=0,\ T = 200$

$200 = 50 + C$

$\therefore T = 50 + 150e^{-kt}$

$t = 1,\ T = 150$

$150 = 50 + 150e^{-k}$

$100 = 150e^{-k}$

$\frac{2}{3} = e^{-k}$

$\ln(2/3) = -k$ (4)

$k = 0.40547$

$T = 90$

$90 = 50 + 150e^{-0.40547t}$

$40 = 150e^{-0.40547t}$

$\frac{4}{15} = e^{-0.40547t}$

$\ln(4/15) = -0.40547t$

$t = 3$ mins 16 sec.

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d) i) $\frac{d^2x}{dt^2} = \frac{dv}{dt}$

$= \frac{dv}{dx} \times \frac{dx}{dt}$

$= \frac{dv}{dx} \times v$ (1)

$= \frac{dv}{dx}\left(\frac{d}{dv}\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

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ii) $v^2 = 2\int a\,dx$

$= 2\int 2x^3 + 4x\,dx$

$= \int 4x^3 + 8x\,dx$

$v^2 = x^4 + 4x^2 + c$

$x = 2,\ v = 6$

$36 = 16 + 16 + c$ (3)

$c = 4$

$\therefore v^2 = x^4 + 4x^2 + 4$

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iii) $v^2 = x^4 + 4x^2 + 4$

$v^2 = (x^2+2)^2$

min speed when $x = 2$

$(2^2+2)^2$ is a min

ie when $x = 2$ (2)

$\therefore v^2 = 36$

$v = 6$

$\therefore$ min speed $= 6$ m/

Ext 1. Assess Task 3 Solns

a) $\int \frac{1}{\sqrt{x^2+1}}\,dx = \ln\left|x+\sqrt{x^2+1}\right| + c$ (1)

b) $\int \sin^2 x \cos x\,dx = \frac{\sin^3 x}{3} + c$ (1)

c) $\int \frac{dx}{x\ln x}$. Let $u = \ln x$, $du = \frac{1}{x}dx$

$\therefore I = \int \frac{du}{u}$

$= \ln u + c$

$= \ln|\ln|x|| + c$ (2)

d) Let $u^2 = x+1$, $2u\,du = dx$. If $x=0$, $u=1$; $x=3$, $u=2$

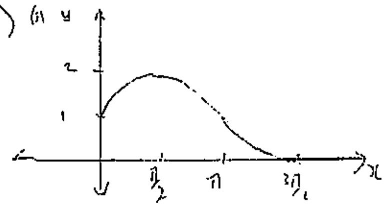
$\therefore I = \int_1^2 \frac{u^2+1}{u}\cdot 2u\,du$

$= 2\left[\frac{u^3}{3}+u\right]_1^2$

$= 2\left\{\left(\frac{8}{3}+2\right)-\left(\frac{1}{3}+1\right)\right\}$

$= \frac{20}{3}$. (4)

e) (i)



(2)

(ii) $V = \pi\int_0^{3\pi/2} y^2\,dx$

$= \pi\int_0^{3\pi/2} 1 + 2\sin x + \sin^2 x\,dx$

$= \pi\int_0^{3\pi/2} 1 + 2\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x\,dx$

$= \pi\int_0^{3\pi/2} \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x\,dx$

$= \pi\left[\frac{3x}{2} - 2\cos x - \frac{1}{4}\sin 2x\right]_0^{3\pi/2}$

$= \pi\left[\left(\frac{9\pi}{4} - 0 - 0\right) - (0-2-0)\right]$

$= \pi\left(\frac{9\pi}{4}+2\right)$

$= \frac{\pi}{4}(9\pi+8)$ units^3 (3)

f) Let $x = u + \frac{1}{2}$, $dx = du$

$\therefore 1-x = 1-\left(u+\frac{1}{2}\right) = \frac{1}{2}-u$

$\therefore I = \int \frac{du}{\sqrt{\left(\frac{1}{2}+u\right)\left(\frac{1}{2}-u\right)}}$

$= \int \frac{du}{\sqrt{\frac{1}{4}-u^2}}$

$= \sin^{-1} 2u + c$ (3)

$= \sin^{-1} 2\left(x-\frac{1}{2}\right) + c$

$= \sin^{-1}(2x-1) + c$

g) (i) $4\theta = 2n\pi \pm \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= 2n\pi \pm \frac{5\pi}{6}$

$\therefore \theta = \frac{n\pi}{2} \pm \frac{5\pi}{24}$ (2)

(ii) If $0 \le \theta \le 2\pi$

$\theta = \frac{5\pi}{24}, \frac{7\pi}{24}, \frac{17\pi}{24}, \frac{19\pi}{24}, \frac{29\pi}{24}, \frac{31\pi}{24}, \frac{41\pi}{24}, \frac{43\pi}{24}$ (2)

/20.

2010 Maths Ext 1 Assess #3 Solutions

Q2 (a) $f(x) = 2\sin^{-1}(3x)$

(i) $D: -\frac{1}{3} \le x \le \frac{1}{3}$

$R: -\pi \le y \le \pi$ (2)

(ii) $f\left(\frac{1}{6}\right) = 2\sin^{-1}\left(\frac{1}{2}\right)$

$= 2\times\frac{\pi}{6}$

$= \frac{\pi}{3}$ (1)

(iii) $f'(x) = 2\times\frac{1}{\sqrt{1-(3x)^2}}\times 3$

$= \frac{6}{\sqrt{1-9x^2}}$ (or $\frac{2}{\sqrt{\frac{1}{9}-x^2}}$)

$f'\left(\frac{1}{6}\right) = \frac{6}{\sqrt{1-\frac{1}{4}}} = \frac{6}{\sqrt{\frac{3}{4}}}$

$= \frac{12}{\sqrt{3}}$ (2)

$= 4\sqrt{3}$ as req.

(b) $y = \int\frac{dx}{1+x^2}$

$y = \tan^{-1}x + c$

$(x=1, y=0)$ $0 = \frac{\pi}{4} + c$

$\therefore c = -\frac{\pi}{4}$

$\therefore y = \tan^{-1}x - \frac{\pi}{4}$ (3)

$(x=\sqrt{3})$ $y = \tan^{-1}\sqrt{3} - \frac{\pi}{4}$

$= \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$

(c) $\int\frac{dx}{\sqrt{9-16x^2}} = \int\frac{dx}{\sqrt{16\left(\frac{9}{16}-x^2\right)}}$

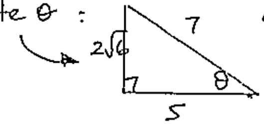
$= \frac{1}{4}\int\frac{dx}{\sqrt{\frac{9}{16}-x^2}}$

$= \frac{1}{4}\cdot\sin^{-1}\frac{4x}{3} + c$ (2)

(d) Let $\cos^{-1}\left(-\frac{5}{7}\right) = \theta$

$\therefore \theta$ is in 2nd Quad.

Acute θ:



and $\cos\theta = -\frac{5}{7}$

$\sin\theta = \frac{2\sqrt{6}}{7}$

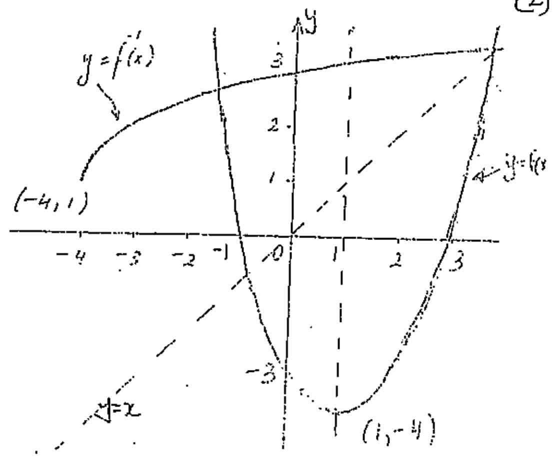
$\therefore \sin\left[2\cos\left(-\frac{5}{7}\right)\right] = \sin 2\theta$

$= 2\sin\theta\cos\theta$

$= 2\times\frac{2\sqrt{6}}{7}\times -\frac{5}{7}$

$= -\frac{20\sqrt{6}}{49}$ (or $-\frac{10\sqrt{24}}{49}$) (3)

(e) (i) $y = (x-3)(x+1)$ & (iv) (2) (2)



(ii) $x \ge 1$ (1)

(iii) $y = x^2 - 2x - 3$

$x = y^2 - 2y - 3$

$x + 4 = y^2 - 2y + 1$

$x + 4 = (y-1)^2$

$\pm\sqrt{x+4} = y - 1$

$\therefore y = 1 \pm \sqrt{x+4}$

Since range is $y \ge 1$

$y = 1 + \sqrt{x+4}$ (3)