Student Name/Number ....

YEAR 12 ASSESSMENT TASK 3

June 23rd 2011

MATHEMATICS
EXTENSION 1.

General Instructions

- Reading Time – 5 minutes
- Working Time – 60 Minutes.
- A table of standard integrals is provided at the back of this paper.
- Start each question on a new page.
- All necessary working should be shown in every question.

QUESTION	MARK
1	
2	
3	
Total /45	

Instructions: **Answer each question on a new page**

Question 1 (15 marks) Marks

a) Find the exact value of $\cos\left(\sin^{-1}\frac{40}{41}\right)$ (2)

b) i) Find $\frac{d}{dx}\sin^{-1}\left(\frac{4x}{3}\right)$. (2)

ii) Hence calculate $\int_0^{\frac{3}{4}}\left(9-16x^2\right)^{\frac{-1}{2}}dx$ (2)

c) i) Given that $x^2+4x+5\equiv(x+a)^2+b^2$, find a and b (2)

ii) Hence $\int\frac{1}{x^2+4x+5}\,dx$ (1)

d) Differentiate $y=\cos^{-1}\sqrt{3x^2-1}$ (2)

e) If $f(x)=2\sin^{-1}3x$ find

i) the domain and range of $f(x)$ (2)

ii) Sketch $f(x)=2\sin^{-1}3x$ (2)

QUESTION 2. (16 Marks) Answer on a new page.

a) N is the number of kangaroos in a certain population at time t years. The population size N satisfies the equation

$$\frac{dN}{dt} = -k(N-500) \text{ for some constant } k.$$

i) Verify that $N = 500 + Ae^{-kt}$ where A is a constant, is a solution of the equation (2)

ii) Initially, there are 3500 kangaroos but after 3 years there are only 3300 left. Find the value of A and the exact value of k. (2)

iii) Find when the number of kangaroos begins to fall below 2300. (2)

iv) Sketch the graph of the population size against time. (2)

b) The speed v *m/s* of a particle moving in a straight line is given

by $v^2 = 64\text{-}16x\text{-}8x^2$ where the displacement from a fixed point O

is x metres.

i) Find an expression for the acceleration and show the motion is simple harmonic. (3)

ii) Between which two points is the particle oscillating? (2)

iii) Find the period and amplitude of the motion. (2)

iv) Find the maximum speed of the particle (1)

QUESTION 3. (14 Marks) Answer on a new page.

a) If $\frac{dx}{dt} = x + 6$ and $x = -5$ when $t = 0$, find an expression for x in terms of t. (3)

b) The volume of a cube is expanding at the constant rate of $5mm^3/\sec$. At what rate is the surface are of the cube increasing when the side length of the cube is 60 centimetres. (3)

c) i) The curve $y = x^4$ is rotated one revolution about the y axis to form a container for storing water. Calculate the volume of water that can be stored if the container is filled to a depth of h metres (3)

ii) Water is poured into the above container at a rate of *60 ml/minute*. Find the rate at which the depth is increasing when the depth is 16cm. (3)

d) The equation of motion of a particle moving along a horizontal straight line is given by the formula $x = 3\cos\left(\frac{1}{4}t\right) + \sin\left(\frac{1}{4}t\right)$ where x is the displacement of the particle at time t seconds.

Explain whether the particle is initially moving to the right or left

and whether it is speeding up or slowing down. (2)

END OF ASSESSMENT TASK.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

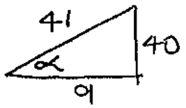
$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

Question 1

a) 

$\cos\alpha = \frac{9}{41}$

b) i) $\frac{1}{\sqrt{1-\left(\frac{4x}{3}\right)^2}} \times \frac{4}{3}$ (1)

$= \frac{3}{\sqrt{9-16x^2}} \times \frac{4}{3}$

$= \frac{4}{\sqrt{9-16x^2}}$

ii) $\frac{1}{4}\int_0^{3/4} \frac{1}{\sqrt{9-16x^2}}\,dx$

$\frac{1}{4}\left[\sin^{-1}\left(\frac{4x}{3}\right)\right]_0^{3/4}$

$\frac{1}{4}\left[\sin^{-1}1 - \sin^{-1}0\right]$

$\frac{1}{4}\times\frac{\pi}{2} = \frac{\pi}{8}$

c) $x^2+4x+5 = x^2+4x+4+1$

$= (x+2)^2 + 1^2$

$a=2 \quad b=\pm 1$ (2)

ii) $\int \frac{1}{(x+2)^2+1^2}\,dx = \tan^{-1}(x+2)+c$ (1)

d) $\frac{-1}{\sqrt{1-(3x^2-1)}} \times \frac{1}{2}(3x^2-1)^{-1/2}\cdot 6x$ $\quad (3x^2-1)^{1/2}$

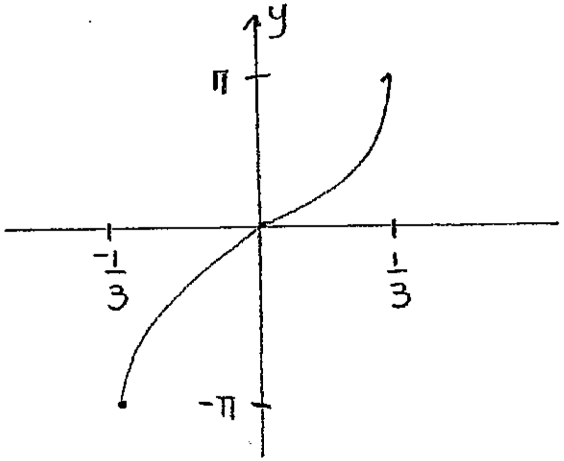
$= \frac{-1}{\sqrt{2-3x^2}} \times \frac{3x}{\sqrt{3x^2-1}}$ or $= \frac{-3x}{\sqrt{2-3x^2}\times\sqrt{3x^2-1}}$ (2)

e) i) $-1 \le 3x \le 1$

$-\frac{1}{3} \le x \le \frac{1}{3}$ (1)

$-\frac{\pi}{2} \le \frac{y}{2} \le \frac{\pi}{2}$

$-\pi \le y \le \pi$ (1)

ii) 

(2)

QUESTION 2

i) $N = 500 + Ae^{-kt}$

$$\frac{dN}{dt} = -kAe^{-kt}$$

But $Ae^{-kt} = N - 500$

$$\therefore \frac{dN}{dt} = -k(N-500)$$

ii) $t=0 \quad N = 3500$

$$3500 = 500 + A$$

$$A = 3000$$

$$3300 = 500 + 3000e^{-3k}$$

$$2800 = 3000e^{-3k}$$

$$\frac{14}{15} = e^{-3k}$$

$$\log_e\left(\frac{14}{15}\right) = -3k$$

$$k = -\frac{1}{3}\log_e\left(\frac{14}{15}\right)$$

iii) $N < 2300$

$$500 + 3000e^{-kt} < 2300$$

$$3000e^{-kt} < 1800$$

$$e^{-kt} < \frac{3}{5}$$

$$\log_e e^{-kt} < \frac{3}{5}$$

$$\frac{1}{3}\log_e\left(\frac{14}{15}\right)t < \log_e 3/5$$

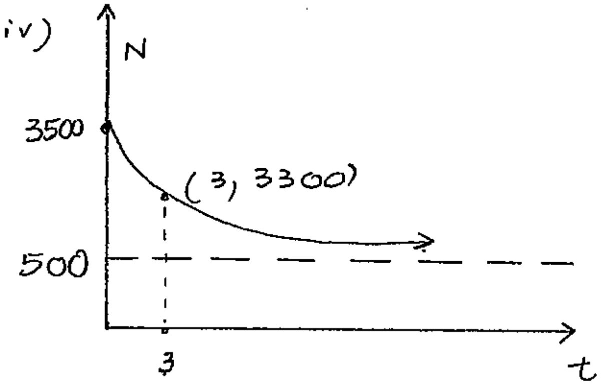
$$t < \frac{\log_e 3/5}{1/3\,\log_e\left(14/15\right)}$$

$$-0.0229t < -0.5108$$

$t > 22.3$ (> 22.21) (2)

allow rounding.

iv)



b) $\frac{1}{2}v^2 = 32 - 8x - 4x^2$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -8 - 8x$$

$$\ddot{x} = -8(1+x)$$

$$\therefore \ddot{x} = -n^2(X)$$

$\therefore$ SHM centre $x = -1$

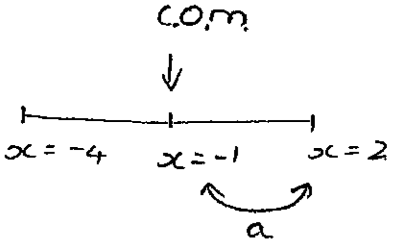
ii) let $v^2=0$ (at rest)

$0 = 64-16x-8x^2$

$0 = x^2+2x-8$

$0 = (x+4)(x-2)$ (2)

$x=-4$ $x=2$



iii) $T = \frac{2\pi}{n}$

$= \frac{2\pi}{\sqrt{8}}$ or $\frac{\pi}{\sqrt{2}}$ (1)

$a = 3$ (1)

v) max speed at centre of motion $x=-1$

$v^2 = 64-16\times-1-8\times(-1)^2$

$= 64+16-8$

$= 72$

$v = \sqrt{72}$ or $6\sqrt{2}$ m/s. (1)

QUESTION 3

a)

$\frac{dx}{dt} = x+6$

$\frac{dt}{dx} = \frac{1}{x+6}$

$t = \ln(x+6)+c.$

when $t=0$ $x=-5$

$0 = \ln 1 + c$

$\therefore c=0$

$t = \ln(x+6)$

$e^t = x+6$

$x = e^t - 6.$

b) $V = x^3$ $\frac{dV}{dt} = 5$

$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}.$

$5 = 3x^2 \cdot \frac{dx}{dt}$

$\frac{5}{3x^2} = \frac{dx}{dt}$

$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$

$= 12x \cdot \frac{5}{3x^2}$

$= \frac{20}{x}$

when $x = 60$ cm $= 600$mm

$\frac{dA}{dt} = \frac{20}{600} = \frac{1}{30}$ mm^2/sec

c) i) $V = \pi \int_0 \sqrt{y}\ dy$

$= \pi \int_0^h y^{1/2}\, dy$

$= \pi \left[\frac{2}{3} y^{3/2}\right]_0^h = \pi \left[\frac{2}{3} h^{3/2}\right]$

$= \frac{2\pi}{3} h^{3/2}$

y axis

$y = x^4$

$\sqrt{y} = x^2$

i) $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$60 = \pi\sqrt{h} \cdot \frac{dh}{dt}$

$\frac{60}{\pi\sqrt{h}} = \frac{dh}{dt}$

$\frac{15}{\pi} = \frac{dh}{dt}.$

$\frac{dV}{dh} = \frac{3}{2} \times \frac{2}{3} \pi h^{1/2}$

$= \pi\sqrt{h}$

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d) $\dot{x} = \frac{-3}{4} \sin \frac{t}{4} + \frac{1}{4} \cos\left(\frac{t}{4}\right)$

$\ddot{x} = \frac{-3}{16} \cos \frac{t}{4} - \frac{1}{16} \sin\left(\frac{t}{4}\right)$

$t = 0$ $v = \frac{1}{4}$, $\ddot{x} = \frac{-3}{16}$

$v > 0$ particle moving to right

$\ddot{x} < 0$ slowing down