Name______________________________ Teacher________________________

GOSFORD HIGH SCHOOL

2012

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 3

MATHEMATICS – EXTENSION 1

Duration- 60 minutes plus 5 minutes reading time

Special Instructions - Tear off back page (**multiple choice response/standard integrals**)

- **Each section** will be collected **separately**

Section 1 **Multiple choice**	**4** questions worth **1 mark each.** (Answer this section on the **multiple choice response sheet provided)**	/4
Section 2 **Inverse Functions**	Answer this section on your **own paper.**	/9
Section 3 **Inverse Trig Functions**	Answer this section on your **own paper.**	/9
Section 4 **General Solutions**	Answer this section on your **own paper.**	/9
Section 5 **Integration involving Trigonometric Functions**	Answer this section on your **own paper.**	/9
TOTAL		/40

Section 1 Multiple Choice (Total 4 marks)

Answer on the multiple choice response sheet provided. Each question is worth 1 mark.

Question 1

The **range** of $y = a + b\tan^{-1}(x-c)$ where *a, b* and *c* are positive real constants is

A. $\frac{-\pi}{2} < y < \frac{\pi}{2}$

B. $a - \frac{c\pi}{2} < y < a + \frac{c\pi}{2}$

C. $c - \frac{b\pi}{2} < y < c + \frac{b\pi}{2}$

D. $a - \frac{b\pi}{2} < y < a + \frac{b\pi}{2}$

Question 2

The curve given by $y = \sin^{-1}(2x)$ where $0 \le x \le \frac{1}{2}$, is rotated about the y-axis to form a solid of revolution. The volume of the solid may be found by evaluating

A. $\frac{\pi}{4}\int_0^{\frac{\pi}{2}}(1-\cos(2y))\,dy$

B. $\frac{\pi}{8}\int_0^{\frac{1}{2}}(1-\cos(2y))\,dy$

C. $\frac{\pi}{8}\int_0^{\frac{\pi}{2}}(1-\cos(2y))\,dy$

D. $\frac{1}{8}\int_0^{\frac{\pi}{2}}(1-\cos(2y))\,dy$

Question 3

Consider the function $f(x) = \frac{1}{1+x^2}$ where $x \geq 0$.

Which of the following **best** represents the **inverse** function for $f(x)$?

A. $f^{-1}(x) = \frac{1}{1+y^2}\ for\ y \geq 0$

B. $f^{-1}(x) = \frac{1}{\sqrt{x}} - 1\ for\ x \geq 0$

C. $f^{-1}(x) = \pm\sqrt{\frac{1}{x} - 1}\ for\ x > 0$

D. $f^{-1}(x) = \sqrt{\frac{1}{x} - 1}\ for\ x > 0$

Question 4

Which of the following is **NOT** true for the function $y = \sin^{-1} x$

A. It is an odd function

B. It is equivalent to $x = siny$ for all real y

C. It has a domain of $-1 \leq x \leq 1$

D. It is differentiable for the domain $-1 < x < 1$

End of multiple choice section

Section 2 Inverse Functions (Total 9 marks)

a. (i) If $f(x) = e^{x+2}$, find $f^{-1}(x)$ **2**

(ii) On the same axes, neatly sketch the graphs of
$f(x)$ and $f^{-1}(x)$ **2**

b. (i) Explain why the function $f(x) = \sqrt{x-2}$ has an
inverse function $f^{-1}(x)$ **1**

(ii) Write down the domain and range of $f(x)$ and $f^{-1}(x)$ **2**

c. Show that the following pairs of functions
$f(x) = 2x - 1$ *and* $g(x) = \frac{x+1}{2}$ are inverses
by showing that $f[g(x)] = g[f(x)] = x$ **2**

Start a new page

Section 3 Inverse Trigonometry (Total 9 marks)

a. Determine the exact value of $\cos\left\{2\sin^{-1}(\frac{12}{13})\right\}$ **2**

b. Differentiate $y = \ln(\cos^{-1} x)$ **2**

c. (i) Write down the result for $\tan 2x$ in terms of $\tan x$ **1**

(ii) Using the fact that $\tan\frac{\pi}{4} = 1$ and considering the above result,
Show that $\tan\frac{\pi}{8} = \sqrt{2} - 1$ **2**

(iii) By consideration of the graph $y = \tan^{-1}(x-1)$ and using the result from part (ii), find the minimum value of a *(where a is a constant)* for which $\tan^{-1}(x-1) + a\tan\frac{\pi}{8} > 0$ for all x. *2*

Start a new page

Section 4 General Solutions to Trigonometric Equations (Total 9 marks)

a. Find the general solutions of $\tan\theta = 1$ **1**

b. Find the general solutions to $sin^2 2x = \frac{3}{4}$ **2**

c. Find the general solutions to $3\,cos\theta - \sqrt{3}\,sin\theta + 3 = 0$ **3**

d. Write down the general solutions to $cos2\theta = sin\theta$ **3**

Start a new page

Section 5 Integration involving Trigonometric Functions (Total 9 marks)

a. Use the table of standard integrals to find the exact value of **2**

$$\int_0^{\frac{\pi}{6}} sec4x\ tan4x\ dx$$

b. Find $\int cos^2 2x\ dx$ **3**

c. Find $\int tan^2 2x\ dx$ **1**

d. Evaluate **3**

$$\int_{\frac{-1}{2}}^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\ dx \quad (using\ the\ substitution\ x = cos\theta)$$

END OF EXAMINATION

Name:________________________ Teacher:____________________

Multiple-choice answer sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely, using a black pen.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A B C D

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A B C D

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A B C D

Start here →

1. A B C D
2. A B C D
3. A B C D
4. A B C D

FINAL ANSWERS TO YR12 ext 1 task 3
2012

Section 1

Q1/ $-\frac{\pi}{2} \le \frac{y-a}{b} < \frac{\pi}{2}$

$a - \frac{b\pi}{2} < y < a + \frac{b\pi}{2}$ (D)

Q2/ $2x = \sin y$

$x = \frac{1}{2}\sin y$

$x^2 = \frac{1}{4}\sin^2 y$

$\cos 2y = 1 - 2\sin^2 y$

$\frac{1 - \cos 2y}{2} = \sin^2 y$

when $x = 0$
$y = 0$

when $x = \frac{1}{2}$
$y = \sin^{-1} 1$
$= \frac{\pi}{2}$

$\frac{\pi}{8}\int_0^{\frac{\pi}{2}} (1 - \cos 2y)\, dy$

(C)

Q3/ $x = \frac{1}{1+y^2}$

$1 + y^2 = \frac{1}{x}$

$y^2 = \frac{1}{x} - 1$

$y = \pm\sqrt{\frac{1}{x} - 1}$

For $f(x)$ D: $x \ge 0$
R: $y > 0$

For $f^{-1}(x)$ D: $x > 0$
R: $y \ge 0$

$\therefore y = \sqrt{\frac{1}{x} - 1}$ for $x > 0$

(D)

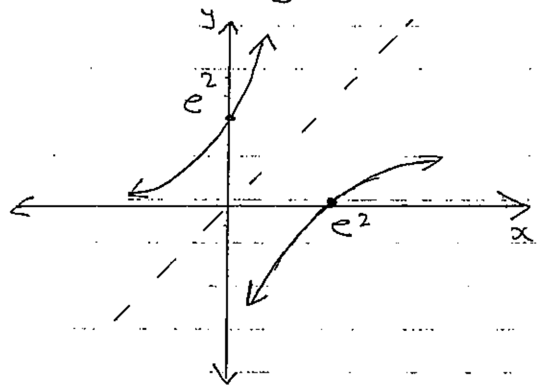
Q4/ (B)

Section 2

(a) $x = e^{y+2}$

$\ln x = y + 2$

$\ln x - 2 = y$ $\therefore f^{-1}(x) = \ln x - 2$



(b) (i) For $f(x)$, for every x value there is only one $f(x)$ value and for every $f(x)$ value there is only one x value.

(ii) For $f(x)$ D: $x \ge 2$
R: $y \ge 0$

For $f^{-1}(x)$ D: $x \ge 0$
R: $y \ge 2$

(c) $f[g(x)] = 2\left(\frac{x+1}{2}\right) - 1$

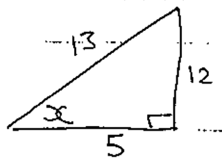
$= x + 1 - 1$

$= x$

$$g[f(x)] = \frac{(2x-1)+1}{2}$$
$$= \frac{2x}{2}$$
$$= x$$

Section 3

(a) Let $x = \sin^{-1}\left(\frac{-12}{13}\right)$



$$\therefore \cos 2x = \cos^2 x - \sin^2 x$$
$$= \left(\frac{5}{13}\right)^2 - \left(\frac{-12}{13}\right)^2$$
$$= \frac{25}{169} - \frac{144}{169}$$
$$= \frac{-119}{169}$$

(b) $y = \ln(\cos^{-1} x)$

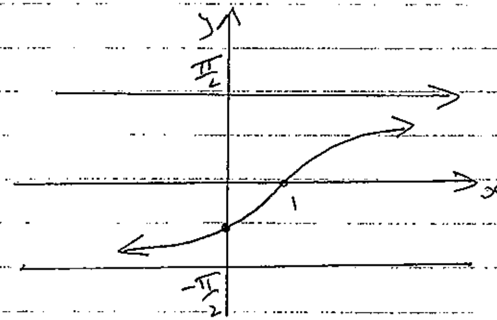
$$y' = \frac{-1}{\sqrt{1-x^2}} \div \cos^{-1} x$$
$$= \frac{-1}{\cos^{-1} x \sqrt{1-x^2}}$$

(c) (i) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(ii) $\tan\dfrac{\pi}{4} = \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

$$1 - \tan^2\frac{\pi}{8} = 2\tan\frac{\pi}{8}$$
$$0 = \tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1$$
$$\tan\frac{\pi}{8} = \frac{-2 \pm \sqrt{4 - 4\times 1\times -1}}{2}$$
$$= \frac{-2 \pm \sqrt{8}}{2}$$
$$= \frac{-2 \pm 2\sqrt{2}}{2}$$
$$= -1 \pm \sqrt{2} \quad \text{but } \tan\frac{\pi}{8} > 0$$
$$\therefore \tan\frac{\pi}{8} = -1 + \sqrt{2}$$

(iii)



$$a\tan\frac{\pi}{8} = \frac{\pi}{2}$$
$$a(\sqrt{2}-1) = \frac{\pi}{2}$$
$$a = \frac{\pi}{2\sqrt{2}-2}$$

Section 4

(a) $\theta = n\pi + \tan^{-1} 1$

$= n\pi + \frac{\pi}{4}$

(b) $\sin 2x = \pm\frac{\sqrt{3}}{2}$

$2x = n\pi + (-1)^n \sin^{-1} \pm\frac{\sqrt{3}}{2}$

$2x = n\pi + (-1)^n \cdot \left(\pm \frac{\pi}{3}\right)$

$x = \frac{n\pi}{2} + (-1)^n \cdot \left(\pm\frac{\pi}{6}\right)$

(c) $3\cos\theta - \sqrt{3}\sin\theta = -3$

$R = \sqrt{9+3}$
$= \sqrt{12}$
$= 2\sqrt{3}$

$\tan\alpha = \frac{\sqrt{3}}{3}$

$\alpha = \frac{\pi}{6}$

$2\sqrt{3}\cos\left(\theta + \frac{\pi}{6}\right) = -3$

$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{-3}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{-3\sqrt{3}}{6}$

$= -\frac{\sqrt{3}}{2}$

$\theta + \frac{\pi}{6} = 2n\pi \pm \frac{5\pi}{6}$

$\theta = 2n\pi \pm \frac{5\pi}{6} - \frac{\pi}{6}$

$\theta = 2n\pi + \frac{2\pi}{3}$ OR

$\theta = 2n\pi - \pi$

(d) $\cos 2\theta = \sin\theta$

$1 - 2\sin^2\theta = \sin\theta$

$0 = 2\sin^2\theta + \sin\theta - 1$

$2\sin\theta$ ✕ -1
$\sin\theta$ ✕ $+1$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{2}$ OR $\sin\theta = -1$

$\theta = n\pi + (-1)^n \cdot \frac{\pi}{6}$ OR $n\pi + (-1)^n \cdot \left(-\frac{\pi}{2}\right)$

Section 5

(a) $\frac{1}{4}\left[\sec 4x\right]_0^{\frac{\pi}{6}} = \frac{1}{4}\left[\frac{1}{\cos\frac{2\pi}{3}} - \frac{1}{\cos 0}\right]$

$= \frac{1}{4} \times \left[\frac{1}{-\frac{1}{2}} - \frac{1}{1}\right]$

$= \frac{1}{4} \times \left[-2 - 1\right]$

$= -\frac{3}{4}$

(b) $\cos 4x = \cos^2 2x - \sin^2 2x$

$= 2\cos^2 2x - 1$

$\cos 4x + 1 = 2\cos^2 2x$

$\therefore \cos^2 2x = \frac{1}{2}\left[\cos 4x + 1\right]$

$$\therefore = \frac{1}{2}\int \cos 4x + 1 \; dx$$

$$= \frac{1}{2}\left[\frac{1}{4}\sin 4x + x\right] + C$$

$$= \frac{\sin 4x}{8} + \frac{x}{2} + C$$

(c) $$\int \sec^2 2x - 1 \; dx = \frac{1}{2}\tan 2x - x + c$$

(d) $x = \cos\theta$

$\frac{dx}{d\theta} = -\sin\theta$

$dx = -\sin\theta \, d\theta$

When $x = \frac{1}{2}$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta = \frac{\pi}{3}$

When $x = -\frac{1}{2}$

$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$

$\theta = \frac{2\pi}{3}$

$$\therefore = \int_{\frac{2\pi}{3}}^{\frac{\pi}{3}} \frac{-\sin\theta\cos\theta}{\sqrt{\sin^2\theta}} \, d\theta$$

$$= -\int_{\frac{2\pi}{3}}^{\frac{\pi}{3}} \cos\theta \; d\theta$$

$$= \left[-\sin\theta\right]_{\frac{2\pi}{3}}^{\frac{\pi}{3}}$$

$$= -\sin\frac{\pi}{3} + \sin\frac{2\pi}{3}$$

$$= -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$$

$$= 0$$

NB// If students recognise it is an odd function and give a bold answer of zero, they have not shown the substitution