GOSFORD HIGH SCHOOL

2013

HSC ASSESSMENT TASK 3

EXTENSION 1 MATHEMATICS

General Instructions:

Total marks: - 40

Attempt all Questions

- Working time: 60 minutes
- Write using black or blue pen.
- Board-approved calculators may be used.
- Each question in Section 2 should be Started on a separate page.
- All necessary working should be shown in every question.

MULTIPLE CHOICE	**/ 4**
INVERSE FUNCTIONS	**/ 24**
GENERALSOLUTIONS	**/ 3**
FURTHER RATES OF CHANGE	**/ 9**
TOTAL	**/ 40**

Name:______________________ Teacher:________________

Multiple-choice answer sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely, using a black pen.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct ↓

A ⊗ B ⊗ C ○ D ○

Start here →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○

MULTIPLE CHOICE

(Answer on the sheet provided)

1) The exact value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is

A) $-\frac{\pi}{6}$ **B)** $\frac{5\pi}{6}$

C) $-\frac{\pi}{3}$ **D)** $\frac{2\pi}{3}$

2) $f(x) = 3x + 2$ for $x \geq -2$

Which of the following represents $f^{-1}(x)$

A) $f^{-1}(x) = \frac{x-2}{3}$ for $x \geq -2$ **B)** $f^{-1}(x) = \frac{2-x}{3}$ for $x \geq -2$

C) $f^{-1}(x) = \frac{x-2}{3}$ for $x \geq -4$ **D)** $f^{-1}(x) = \frac{x-2}{3}$ for $x \leq -4$

3) If $\tan\theta = \sqrt{3}$ then

A) $\theta = 2n\pi \pm \frac{\pi}{3}$ **B)** $\theta = 2n\pi + \frac{\pi}{3}$

C) $\theta = n\pi \pm \frac{\pi}{3}$ **D)** $\theta = n\pi + \frac{\pi}{3}$

4) A square is expanding so that its area is increasing at a constant rate of $12cm^2/\sec$. The rate of increase of the side of the square when the area is $36cm^2$ is:

A) $\frac{1}{6}cm/\sec$ **B)** $\frac{1}{3}cm/\sec$

C) $2\,cm/\sec$ **D)** $1\,cm/\sec$

Question 5. (Start a new page)

a) Draw a neat sketch of $y = 3\sin^{-1}2x$ showing clearly the domain and range. **2**

b) Differentiate

i) $\sin^{-1}2x$ **1**

ii) $\tan^{-1}(x+2)$ **1**

iii) $x^2\sin^{-1}x$ **2**

iv) $x\cos^{-1}x - \sqrt{1-x^2}$ **2**

c) Find the indefinite integrals of the following.

i) $\frac{1}{\sqrt{4-x^2}}$ **1**

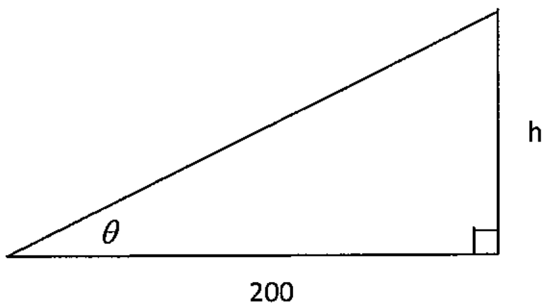
ii) $\frac{1}{9+x^2}$ **1**

iii) $\frac{x+1}{x^2+4}$ **2**

Question 6 (Start a new page)

a) Find all solutions to $\sqrt{3}\cos x + \sin x = 1$ **3**

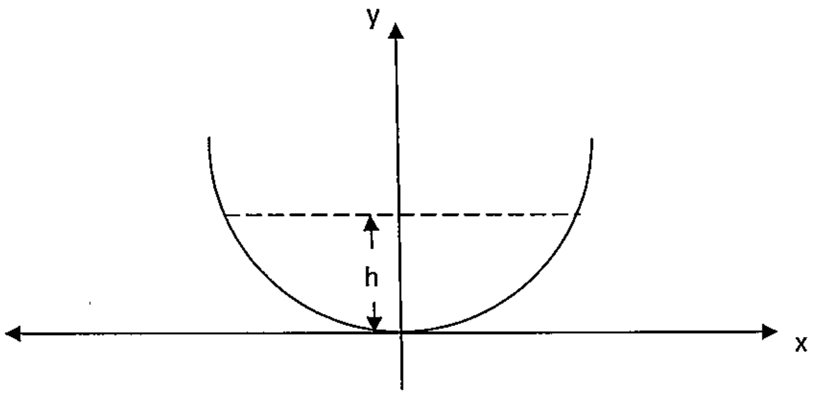
b)



At an air show, a Harrier Jump Jet leaves the ground 200 *metres* from an observer and rises vertically at the rate of 25 *m/sec.* At what rate is the observer's angle of elevation of the aircraft changing when the jet is 500 *metres* above the ground? **3**

QUESTION 6 CONTINUED OVER PAGE

c)



The diagram shows the lower half of the circle whose equation is $x^2 + y^2 - 20y = 0$. A hemispherical bowl is obtained by rotating the semi-circle about the *y*-axis.

i) Show that when the depth of water in the bowl is *h cm.*, the volume of water, *V*, in the bowl is given by:

$$V = \frac{\pi h^2}{3}(30 - h)cm^3$$ **2**

ii) Show that when the depth of water in the bowl is *h cm.*, the surface area of water, *S*, exposed to the open air is given by:

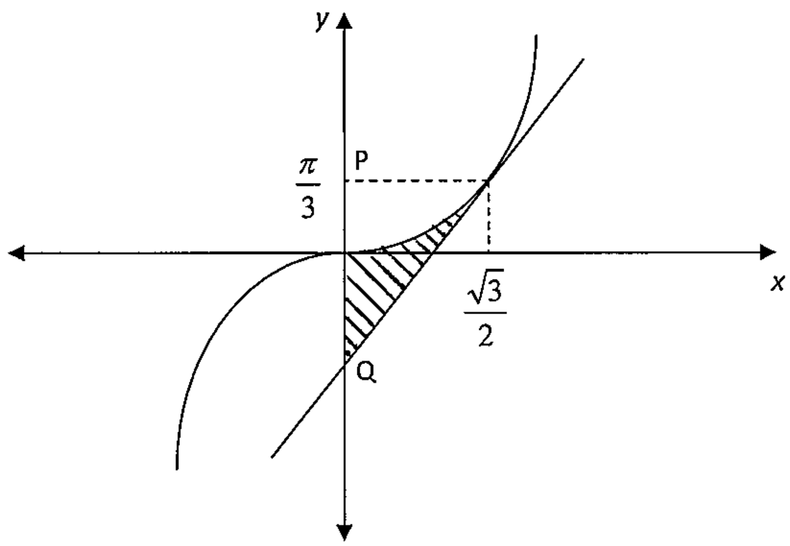
$$S = \pi h(20 - h)cm^2$$ **2**

iii) If water is poured into the bowl at the rate of $50cm^3/\sec$ determine, at the instance when the depth of water is 5*cm* the rate of increase of the open surface of water in $cm^2/\sec$. **2**

Question 7 (Start a new page)

a) Use the substitution $u = x^2$ to evaluate $\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}\,dx$ **3**

b)



The diagram shows the region bounded by $y = \sin^{-1} x$, the 'y' axis and the tangent to the curve at the point $(\frac{\sqrt{3}}{2}, \frac{\pi}{3})$.

Show that the volume of the solid formed when the region is rotated about the 'y' axis is $\frac{\pi}{24}(9\sqrt{3} - 4\pi)$ cubic units. **4**

c) A function is defined by $g(x) = \frac{x+2}{x+1}$, for $x < -1$

i) Find $g^{-1}(x)$ and state its domain and range. **3**

ii) Find any point(s) of intersection for the curves

$\mathrm{y} = \mathrm{g(x)}$ and $\mathrm{y} = g^{-1}(x)$ **2**

END OF THE EXAMINATION

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

EXTENSION 1 ASSESSMENT TASK SOLUTIONS (TASK 3 2013)

MULTIPLE CHOICE

1) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$ (B)

2) $f(x): 3x+2$ D: $x \geq -2$

$\therefore$ R: ≥ -4

$f^{-1}(x)$: $x = 3y+2$

$y = \frac{x-2}{3}$

D: $x \geq -4$

(C)

3) (D)

4) $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

$12 = 2x \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{12}{2x}$

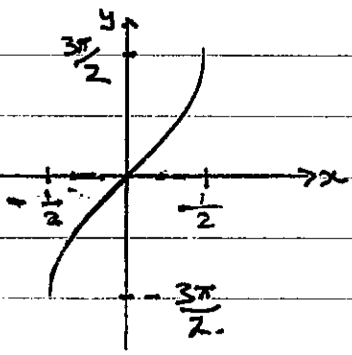
(Area = $36\text{cm}^2 \rightarrow x = 6$)

$\frac{dx}{dt} = \frac{12}{12}$

$= 1$ (D)

Question 5

a) $y = 3\sin^{-1}2x$



b) i) $\frac{d}{dx}\left(\sin^{-1}2x\right) = \frac{2}{\sqrt{1-4x^2}}$

ii) $\frac{d}{dx}\left(\tan^{-1}(x+2)\right) = \frac{1}{1+(x+2)^2}$

iii) $\frac{d}{dx}\left(x^2\sin^{-1}x\right)$

$= \frac{x^2}{\sqrt{1-x^2}} + 2x\sin^{-1}x.$

iv) $\frac{d}{dx}\left(x\cos^{-1}x - \sqrt{1-x^2}\right)$

$= \frac{-x}{\sqrt{1-x^2}} + \cos^{-1}x - \frac{1}{2}\left(\frac{1}{\sqrt{1-x^2}} \times -2x\right)$

$= \frac{-x}{\sqrt{1-x^2}} + \cos^{-1}x + \frac{x}{\sqrt{1-x^2}}$

$= \cos^{-1}x.$

c) i) $\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right) + c$

ii) $\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$

iii) $\int \frac{x+1}{x^2+4}\,dx = \int \frac{x}{x^2+4} + \frac{1}{x^2+4}\,dx$

$= \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

Question 6.

a) $\sqrt{3}\cos x + \sin x = 1$

Let $\sqrt{3}\cos x + \sin x = R\cos(x-\alpha)$

$\therefore \sqrt{3}\cos x + \sin x = R\cos x\cos\alpha + R\sin x\sin\alpha$

equating coefficients

$R\cos\alpha = \sqrt{3}$ ---(1)

$R\sin\alpha = 1$ ---(2)

(2) ÷ (1) $\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

$(1)^2 + (2)^2$ $R^2(\sin^2\alpha + \cos^2\alpha) = 4$

$R^2 = 4$

$R = 2$

$\therefore \sqrt{3}\cos x + \sin x = 2\cos(x - \frac{\pi}{6})$

$\therefore \sqrt{3}\cos x + \sin x = 1$

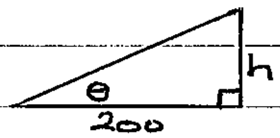
$\Rightarrow 2\cos(x - \pi/6) = 1$

$\cos(x - \pi/6) = \frac{1}{2}$

$x - \pi/6 = 2n\pi \pm \pi/3$

$x = 2n\pi + \pi/2$ OR

$2n\pi - \frac{\pi}{6}$



$\tan\theta = \frac{h}{200}$

$\theta = \tan^{-1}\left(\frac{h}{200}\right)$

$\frac{d\theta}{dh} = \frac{\frac{1}{200}}{1 + \left(\frac{h}{200}\right)^2}$

$= \frac{200}{40000 + h^2}$

Now $\frac{d\theta}{dt} = \frac{d\theta}{dh} \times \frac{dh}{dt}$

$= \frac{200}{40000 + h^2} \times 25$

When $h = 500$

$\frac{d\theta}{dt} = \frac{200}{40000 + 250000} \times 25$

$= \frac{1}{58}{}^c$ /sec.

ALTERNATE METHOD

$\tan\theta = \frac{h}{200}$

$h = 200\tan\theta$

$\frac{dh}{d\theta} = 200\sec^2\theta$

$\frac{dh}{dt} = \frac{dh}{d\theta} \times \frac{d\theta}{dt}$

$25 = 200\sec^2\theta \frac{d\theta}{dt}$

$\therefore \frac{d\theta}{dt} = \frac{25}{200\sec^2\theta}$

$= \frac{1}{8(1 + \tan^2\theta)}$

$= \frac{1}{8\left(1 + \frac{h^2}{200^2}\right)}$

$= \frac{5000}{29000}$

$= \frac{1}{58}{}^c$ /sec.

c) i) $x^2 + y^2 - 20y = 0$

$x^2 = 20y - y^2$

$$V = \pi \int_0^h x^2 \, dy$$

$$= \pi \int_0^h 20y - y^2 \, dy$$

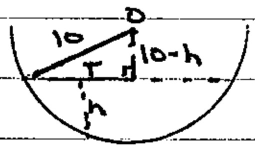
$$= \pi \left[10y^2 - \frac{y^3}{3} \right]_0^h$$

$$= \pi \left[\left(10h^2 - \frac{h^3}{3}\right) - 0 \right]$$

$$= \pi \left[\frac{30h^2 - h^3}{3} \right]$$

$$= \frac{\pi h^2}{3}(30 - h) \text{ cm}^3.$$

ii)



$x^2 + y^2 - 20y + 100 = 100$

$x + (y-10)^2 = 100$

$\therefore$ radius $= 10$

$$r^2 = 10^2 - (10-h)^2$$

$$= 100 - 100 + 20h - h^2$$

$$= 20h - h^2$$

$$A = \pi r^2$$

$$= \pi(20h - h^2)$$

iii) $$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$50 = (20\pi h - \pi h^2) \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{50}{20\pi h - \pi h^2}$$

$h = 5$

$$\frac{dh}{dt} = \frac{50}{75\pi}$$

$$= \frac{2}{3\pi}$$

Now $$\frac{dS}{dt} = \frac{dS}{dh} \times \frac{dh}{dt}$$

$$= (20\pi - 2\pi h) \times \frac{2}{3\pi}$$

$h = 5$: $$\frac{dS}{dt} = 10\pi \times \frac{2}{3\pi}$$

$$= \frac{20}{3} \text{ cm}^2/\text{sec}.$$

Question 7

a) $$\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}} \, dx$$

$u = x^2$ $\qquad x = 0;\ u = 0$

$\frac{du}{dx} = 2x$ $\qquad x = \frac{1}{\sqrt{2}};\ u = \frac{1}{2}$

$du = 2x \, dx$

$$= \frac{1}{2} \int_0^{1/\sqrt{2}} \frac{2x \, dx}{\sqrt{1-x^4}}$$

$$= \frac{1}{2} \int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}}$$

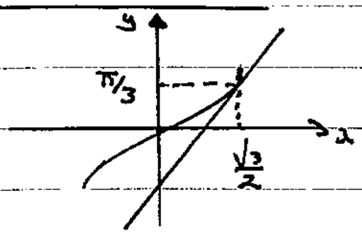
$= \frac{1}{2}\left[\sin^{-1}u\right]_0^{1/2}$

$= \frac{1}{2}\left(\sin^{-1}\frac{1}{2} - \sin^{-1}0\right]$

$= \frac{1}{2}\left(\frac{\pi}{6} - 0\right)$

$= \frac{\pi}{12}.$

b)



$y = \sin^{-1}x$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

at $x = \frac{\sqrt{3}}{2}$: $\frac{dy}{dx} = \frac{1}{\sqrt{1 - 3/4}}$

$= \frac{1}{\sqrt{1/4}}$

$= 2$

∴ equation of the tangent

$y - \frac{\pi}{3} = 2\left(x - \frac{\sqrt{3}}{2}\right)$

$y - \frac{\pi}{3} = 2x - \sqrt{3}$

$y = 2x + \frac{\pi}{3} - \sqrt{3}$

cuts the 'y' axis at Q ie $x = 0$

$y = \pi/3 - \sqrt{3}$

∴ $Q\ (0, \pi/3 - \sqrt{3})$

∴ $PQ = \sqrt{0^2 + (\pi/3 - \sqrt{3} - \pi/3)^2}$

$= \sqrt{3}.$

Now V = Volume of a cone $- \pi\int_c^d x^2\,dy.$

$V = \frac{1}{3} \times \pi \times \left(\frac{\sqrt{3}}{2}\right)^2 \times \sqrt{3} - \pi\int_0^{\pi/3} \sin^2 y\,dy$

$= \frac{\pi\sqrt{3}}{4} - \frac{\pi}{2}\int_0^{\pi/3} 1 - \cos 2y\,dy$

$= \frac{\pi\sqrt{3}}{4} - \frac{\pi}{2}\left[y - \frac{\sin 2y}{2}\right]_0^{\pi/3}.$

$= \frac{\pi\sqrt{3}}{4} - \frac{\pi}{2}\left[\left(\frac{\pi}{3} - \frac{\sin\frac{2\pi}{3}}{2}\right) - 0\right]$

$= \frac{\pi\sqrt{3}}{4} - \frac{\pi}{2}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]$

$= \frac{\pi\sqrt{3}}{4} - \frac{\pi^2}{6} + \frac{\pi\sqrt{3}}{8}$

$= \frac{3\pi\sqrt{3}}{8} - \frac{\pi^2}{6}$

$= \frac{9\pi\sqrt{3}}{24} - \frac{4\pi^2}{24}$

$= \frac{\pi}{24}\left(9\sqrt{3} - 4\pi\right)$ cubic units

c)(i) $g(x) = \frac{x+2}{x+1}$ D: $x < -1$ R: $y < 1$

$g^{-1}(x)$: $x = \frac{y+2}{y+1}$

$xy + x = y + 2$

$xy - y = 2 - x$

$y(x-1) = 2 - x$

$y = \frac{2-x}{x-1}$

∴ $g^{-1}(x) = \frac{2-x}{x-1}.$

∴ D: $x < 1$ R: $y < -1$

ii) Curves will intersect on the line $y = x$:

∴ $\frac{x+2}{x+1} = x$

$x^2 + x = x + 2$

$x^2 = 2$

$x = \pm\sqrt{2}$

∴ pt. of intersection $(-\sqrt{2}, -\sqrt{2})$

(as domain $g(x) < -1$)