Name____________________________ Teacher______________________

GOSFORD HIGH SCHOOL

2015

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 3

MATHEMATICS – EXTENSION 1

Time Allowed - 60 minutes plus 5 minutes reading time

- Write using a black or blue pen. Black pen is preferred.
- Board approved calculators may be used.
- Answers to Questions 1-4 are to be done on the answer sheet provided.
- Questions 5-7 are to be answered on your own paper. Start each of Questions 5, 6 and 7 on a new page
- Relevant mathematical reasoning and/or calculations must be shown for Questions 5-7.

Multiple choice		/4
Question 5 a) and b)	Applications of Calculus to the Physical World	/9
Question 5 c) Question 6 a)	General Solutions to Trigonometric Equations	/6
Question 6 b) c) Question 7	Inverse Functions and Inverse Trigonometric Functions	/21
TOTAL		**/40**

Answer Questions 1 – 4 on the multiple choice answer sheet provided. Questions are worth 1 mark each.

1) $tan^{-1}\left[tan\frac{2\pi}{3}\right] =$

A) $\frac{\pi}{3}$ B) $\frac{-\pi}{3}$ C) $\frac{2\pi}{3}$ D) $\frac{-2\pi}{3}$

2) If $cos\theta = \frac{1}{2}$, then

A) $\theta = n\pi \pm \frac{\pi}{3}$ B) $\theta = n\pi + (-1)^n\frac{\pi}{3}$

C) $\theta = 2n\pi \pm \frac{\pi}{3}$ D) $\theta = 2n\pi + \frac{\pi}{3}$

3) $\int \frac{dx}{\sqrt{4-x^2}} =$

A) $\sin^{-1}\frac{x}{2} + C$ B) $\sin^{-1}\frac{x}{4} + C$

C) $\frac{1}{2}\sin^{-1}\frac{x}{2} + C$ D) $\frac{1}{2}\sin^{-1}x + C$

4) The number N of animals in a population at time t years is given by $N = 100 + Ae^{kt}$, for $A > 0$ and $k > 0$. Which of the following is the correct differential equation?

A) $\frac{dN}{dt} = k(N + 100)$ B) $\frac{dN}{dt} = -k(N - 100)$

C) $\frac{dN}{dt} = k(N - 100)$ D) $\frac{dN}{dt} = -k(N + 100)$

Answer Questions 5 – 7 on your own paper. Start each question on a new page.

5) a) Water is poured into a conical vessel of height 30 cm and radius 24 cm.

i) Show that the volume of water is given by $V = \frac{16\pi h^3}{75}$, when the depth of water is *h* metres. **(2)**

ii) If the depth of water is increasing at the rate of $\frac{1}{2} cm/min$, find the rate of increase of the volume of water when the depth of the water is 20 cm. **(3)**

b) The rate of growth of the number of sheep on a farm is given by $\frac{dN}{dT} = k(N - 300)$, where *N* is the number of sheep and *t* is the time in years.

i) Show that $N = 300 + Ae^{kt}$ is a solution to the differential equation. **(1)**

ii) If initially there are 400 sheep and two years later there are 700, find the number of sheep on the farm at the end of four years. **(3)**

c) Find the general solution of $\sqrt{3}\sin\theta - \cos\theta = \sqrt{3}$ **(3)**

6) a) Find all values of *x* for which $2\cos\left(3x + \frac{\pi}{6}\right) + \sqrt{3} = 0$ **(3)**

b) Consider the function $f(x) = x^2 - 2x$

i) Find the domain over which the function is monotonic increasing. **(1)**

ii) Find the inverse function $f^{-1}(x)$ over this restricted domain. **(2)**

iii) On the same set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ showing where the graphs intersect over their respective domains. **(2)**

c) Find the area of the region bounded by the curve $y = \frac{1}{2}\sin^{-1} x$, the x axis and the line $x = 1$. **(4)**

7) a) Differentiate $\sin^{-1} 5x$ **(1)**

b) Show that $\tan^{-1}(4) - \tan^{-1}(\frac{3}{5}) = \frac{\pi}{4}$ **(3)**

c) Consider the function $y = 3\cos^{-1}\left(\frac{x}{2}\right)$

i) Draw a neat sketch of the function, clearly showing the domain and range. **(2)**

ii) Find the gradient of the tangent to the curve when $y = \pi$ **(2)**

d) Differentiate $x\tan^{-1} x$ and hence evaluate $\int_0^1 \tan^{-1} x\, dx$ **(4)**

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

EXTENSION 1 HSC TASK 3 2015

SOLUTIONS

1) $\tan^{-1}\left[\tan \frac{2\pi}{3}\right] = \tan^{-1}(-\sqrt{3})$

$= -\tan^{-1}(\sqrt{3})$

$= -\frac{\pi}{3}$ (B)

2) $\cos\theta = \frac{1}{2}$

$\theta = 2n\pi \pm \cos^{-1}\frac{1}{2}$

$= 2n\pi \pm \frac{\pi}{3}$ (C)

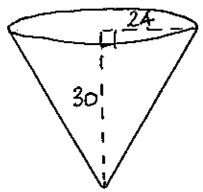
3) $\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2} + C$ (A)

4) $N = 100 + Ae^{kt} \rightarrow Ae^{kt} = N - 100$

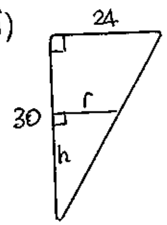
$\frac{dN}{dt} = kAe^{kt}$

$= k(N-100)$ (C)

5) a)



i)



$\frac{r}{24} = \frac{h}{30}$

$r = \frac{24h}{30}$

$= \frac{4h}{5}$

now $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi \cdot \frac{16h^2}{25} h$

$= \frac{16\pi h^3}{75}$

ii) $\frac{dV}{dh} = \frac{48\pi h^2}{75}$ $\qquad \frac{dh}{dt} = \frac{1}{2}$

$= \frac{16\pi h^2}{25}$

now $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$= \frac{16\pi h^2}{25} \cdot \frac{1}{2}$

$= \frac{8\pi h^2}{25}$

When $h = 20$, $\frac{dV}{dt} = \frac{8\pi \times 400}{25}$

$= 128\pi$ cm³/min.

b) $\frac{dN}{dt} = k(N-300)$

i) $N = 300 + Ae^{kt}$

$\frac{dN}{dt} = kAe^{kt}$

$= k(N-300)$ $\quad \therefore N = 300 + Ae^{kt}$ is a sdn

ii) $N = 300 + Ae^{kt}$

when $t = 0$, $N = 400$

$400 = 300 + A$

$A = 100$

$\therefore N = 300 + 100e^{kt}$

when $t = 2$, $N = 700$

$700 = 300 + 100e^{2k}$

$100e^{2k} = 400$

$e^{2k} = 4$

$2k = \ln 4$

$k = \frac{\ln 4}{2}$

$= 0.693$

$\therefore N = 300 + 100e^{0.693t}$

When $t=4$, $N = 300 + 100e^{2.772}$
$= 1899.058$

$\therefore$ 1899 sheep

c) $\sqrt{3}\sin\theta - \cos\theta = \sqrt{3}$

$\sqrt{3}\sin\theta - \cos\theta = R\sin(\theta - \alpha)$
$= R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$R\cos\alpha = \sqrt{3}$ $\quad \tan\alpha = \frac{1}{\sqrt{3}}$ $\quad R = \sqrt{3+1}$
$R\sin\alpha = 1$ $\quad \alpha = \frac{\pi}{6}$ $\quad = 2$

$\therefore \sqrt{3}\sin\theta - \cos\theta = 2\sin\left(\theta - \frac{\pi}{6}\right)$

ie $2\sin\left(\theta - \frac{\pi}{6}\right) = \sqrt{3}$

$\sin\left(\theta - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\theta - \frac{\pi}{6} = n\pi + (-1)^n\frac{\pi}{3}$

$\theta = n\pi + (-1)^n\frac{\pi}{3} + \frac{\pi}{6}$

$\theta = n\pi + \frac{\pi}{3} + \frac{\pi}{6}$ or $\theta = n\pi - \frac{\pi}{3} + \frac{\pi}{6}$
$= n\pi + \frac{\pi}{2}$ $\quad = n\pi - \frac{\pi}{6}$

6) a) $2\cos\left(3x + \frac{\pi}{6}\right) + \sqrt{3} = 0$

$\cos\left(3x + \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$3x + \frac{\pi}{6} = 2n\pi \pm \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$3x + \frac{\pi}{6} = 2n\pi \pm \frac{5\pi}{6}$

$3x = 2n\pi \pm \frac{5\pi}{6} - \frac{\pi}{6}$

$3x = 2n\pi + \frac{5\pi}{6} - \frac{\pi}{6}$ or $3x = 2n\pi - \frac{5\pi}{6} - \frac{\pi}{6}$

$3x = 2n\pi + \frac{2\pi}{3}$ $\quad 3x = 2n\pi - \pi$

$x = \frac{2n\pi}{3} + \frac{2\pi}{9}$ $\quad x = \frac{2n\pi}{3} - \frac{\pi}{3}$

b) $f(x) = x^2 - 2x$

i) $f'(x) = 2x - 2$

for monotonic increasing, $f'(x) > 0$

ie $2x - 2 > 0$
$x > 1$

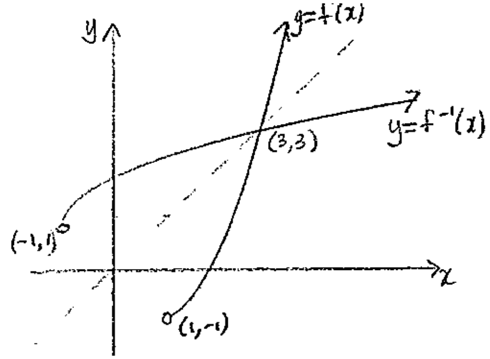
ii) $y = x^2 - 2x$

$x = y^2 - 2y$
$x + 1 = y^2 - 2y + 1$
$x + 1 = (y-1)^2$
$y - 1 = \pm\sqrt{x+1}$
$y = 1 \pm \sqrt{x+1}$

but $y > 1$

$\therefore y = 1 + \sqrt{x+1}$

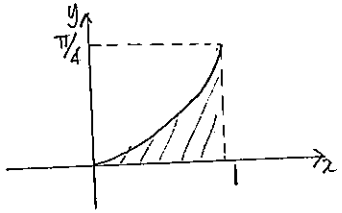
iii)



c) $y = \frac{1}{2}\sin^{-1}x$

$2y = \sin^{-1}x$

$\sin 2y = x$



$$A = \frac{\pi}{4} - \int_0^{\pi/4} \sin 2y \, dy$$

$$= \frac{\pi}{4} - \left[-\frac{1}{2}\cos 2y\right]_0^{\pi/4}$$

$$= \frac{\pi}{4} + \left[\frac{1}{2}\cos\frac{\pi}{2} - \frac{1}{2}\cos 0\right]$$

$$= \frac{\pi}{4} + \left(0 - \frac{1}{2}\right)$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\right) \text{ sq units.}$$

7) a) $\frac{d}{dx}\sin^{-1}5x = \frac{5}{\sqrt{1-25x^2}}$

b) $\tan^{-1}(4) - \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$

let $x = \tan^{-1}(4)$ and $y = \tan^{-1}\left(\frac{3}{5}\right)$

then $\tan x = 4$ then $\tan y = \frac{3}{5}$

now $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

$$= \frac{4 - \frac{3}{5}}{1 + 4 \cdot \frac{3}{5}}$$

$$= \frac{\frac{17}{5}}{\frac{17}{5}}$$

$$= 1$$

$\therefore \tan(x-y) = 1$

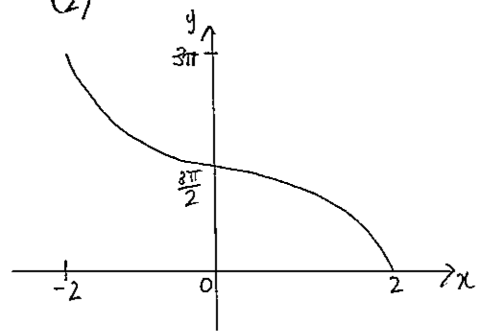
$x - y = \tan^{-1} 1$

$= \frac{\pi}{4}$

$\therefore \tan^{-1}(4) - \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$

c) $y = 3\cos^{-1}\left(\frac{x}{2}\right)$

i)



ii) $y = 3\cos^{-1}\left(\frac{x}{2}\right)$

$y' = \frac{-3}{\sqrt{4-x^2}}$

when $y = \pi$: $3\cos^{-1}\left(\frac{x}{2}\right) = \pi$

$\cos^{-1}\left(\frac{x}{2}\right) = \frac{\pi}{3}$

$\frac{x}{2} = \frac{1}{2}$

$\therefore x = 1$

When $x=1$: $y' = \frac{-3}{\sqrt{4-1}}$

$= \frac{-3}{\sqrt{3}}$

$= -\sqrt{3}$

$\therefore$ gradient is $-\sqrt{3}$.

d) $\frac{d}{dx}\left[x\tan^{-1}x\right] = \tan^{-1}x + x \cdot \frac{1}{1+x^2}$

$= \tan^{-1}x + \frac{x}{1+x^2}$

$\therefore \int_0^1 \tan^{-1}x + \frac{x}{1+x^2}\, dx = \left[x\tan^{-1}x\right]_0^1$

$\int_0^1 \tan^{-1}x\, dx = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$

$= \left[x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right]_0^1$

$= \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) - (0-0)$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$