Year 12 3 UNIT
Term 2 – Assessment - 1999

INSTRUCTIONS:

- Time allowed: 85 minutes
- This is an open book test
- Show all necessary working

QUESTION 1 (START A NEW PAGE)

(a) A container of soup is removed from the top of a campfire. The soup is left to stand for several minutes in surrounds of a constant 15^0. The soup cools at a rate proportional to the difference between its temperature (T) and the surrounding temperature (S), i.e. $\frac{dT}{dt} = -0.05(T-S)$.

(i) Show that $T = S + Ae^{-0.05t}$ is a solution of $\frac{dT}{dt} = -0.05(T-S)$.

(ii) If the soup is initially at 85^0, write down the values of S and A.

(iii) Find the temperature of the soup after 10 minutes. (Give your answer to the nearest degree)

(b) All the letters of the word EQUATION are arranged at random in a line. How many arrangements are possible if

(i) there are no restrictions

(ii) there is a consonant at each end.

(iii) all the vowels are together.

QUESTION 2 (START A NEW PAGE)

(a) The velocity (v ms^{-1}) of an object at position x metres is given by $v = 2\sqrt{x(x-20)}$.

(i) Find the values of x for which motion is possible.

(ii) Show that the acceleration is given by $\ddot{x} = 4(x-10)$.

(iii) Find the velocity when the acceleration is 80 ms^{-2}.

(b) The position of an object at time t is given by $x = \sin^2 6t$. Prove that the acceleration $\ddot{x}$ can be expressed in the form $\ddot{x} = -n^2(x-b)$.

QUESTION 3 (START A NEW PAGE)

(a) A toy rocket is fired from the top of a 100 metre high cliff and its position at time t is given by the equations $x = 30t$ and $y = -5t^2 + 40t + 100$. Find the

(i) greatest height above the ground that the rocket reaches.

(ii) distance from the base of the cliff to the point where the rocket hits the ground.

(b) Three notes are chosen at random from an envelope containing three \$5, five \$10 and two \$20 notes. Find the number of different selections that can be made if

(i) they are all different denominations

(ii) their total value is \$30

QUESTION 4 (START A NEW PAGE)

(a) 100 white and 100 black marbles are mixed together. Some are placed in urn A while the rest are placed in urn B. You are told that the probability of selecting a white marble from urn A is $\frac{2}{3}$ and if a white marble is placed in urn B then the probability of selecting a black marble from urn B is also $\frac{2}{3}$. Find the number of each colour originally in urn A.

(b) An object moving with SHM has its position at time t given by $x = a\cos(nt+\alpha)$. The object is initially 6 metres to the right of the origin. If the period of its motion is 8 minutes and its maximum speed is 3π m/min , find

(i) the values of n, a and α.

(ii) the first time when the object passes through the origin.

QUESTION 5 (START A NEW PAGE)

(a) Four girls (Anne, Betty, Carole and Dianne) and four boys (Peter, Quentin, Ross and Stuart) are seated around a circular table.

(i) Find the number of different random arrangements that are possible

Find the probability that in a random arrangement

(ii) all the girls are seated together.

(iii) the boys and girls sit in alternate seats.

(b) In a colony of 1million birds, the number infected with a disease after time t weeks is given by $N = \dfrac{1000000}{100 + 1900e^{-0.05t}}$.

(i) How many birds are initially infected?

(ii) How many birds will ultimately be infected?

(iii) Show that $\dfrac{dN}{dt} = \dfrac{N(10000 - N)}{200000}$.

(iv) How many birds will be present when the rate of infection is greatest?

infected

QUESTION 6 (START A NEW PAGE)

(a) Show that $\dfrac{d}{dx}\left[\cos^{-1}\left(\dfrac{x^2}{16}\right)\right] = \dfrac{-2x}{\sqrt{256 - x^4}}$.

(b) A particle initially at rest at x = 4, has an acceleration towards the origin given by $\ddot{x} = -4\left(x + \dfrac{256}{x^3}\right)$ for x > 0.

(i) Show that the velocity v is given by $v^2 = 4\left(\dfrac{256}{x^2} - x^2\right)$.

(ii) Explain why the velocity is negative for 0 < x < 4.

(iii) Using part (a) find an expression for the time taken to reach position x.

(iv) Find the time for the particle to reach $x = 2\sqrt{2}$.

THIS IS THE END

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

YEAR 12 1999 3UNIT TERM 2 ASSESSMENT

QUESTION 1

(a)(i) $\frac{dT}{dt} = -0.05Ae^{-0.05t}$

$= -0.05(T-S)$ since $Ae^{-0.05t} = T-S$

(ii) $S = 15$

$t = 0,\ T = 85$

$\therefore 85 = 15 + Ae^{0}$

$A = 70$

$\therefore A = 70,\ S = 15$

(iii) $T = 15 + 70e^{-0.05t}$

$t = 10,\ T = 15 + 70e^{-0.5}$

$= 57.46°$

$= 57°$ (to nearest degree)

(b)(i) N° arrangements $= 8!$

(ii) Arrange consonant $= 3.2$

arrange rest $= 6!$

Total $= 3.2.6!$

(iii) arrange vowels $= 5!$

arrange vowel group + 3 others $= 4!$

Total $= 5!.4!$

QUESTION 2

(a)(i) $x(x-20) \geq 0$

$\therefore x \leq 0,\ x \geq 20$

(ii) $\ddot{x} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$ $\qquad \tfrac{1}{2}v^2 = 2x(x-20)$

$= \frac{d}{dx}(2x^2 - 40x)$

$= 4x - 40$

$\ddot{x} = 4(x-10)$

(iii) $\ddot{x} = 80 \Rightarrow 80 = 4(x-10)$

$20 = x - 10$

$x = 30$

$\therefore V = 2\sqrt{30.10}$

$= 20\sqrt{3}$ m/s

(b) $x = \sin^2 6t$

$\dot{x} = 2\sin 6t \cdot 6\cos 6t$

$= 12\sin 6t\cos 6t$

$= 6\sin 12t$

$\ddot{x} = 72\cos 12t$

$= 72(1 - 2\sin^2 6t)$

$= 72(1-2x)$

$= -144(x - \tfrac{1}{2})$

or

$\dot{x} = 12\sin 6t\cos 6t$

$\ddot{x} = 12\{(\sin 6t)(-6\sin 6t) + (6\cos 6t)(\cos 6t)\}$

$= 12\{-6\sin^2 6t + 6\cos^2 6t\}$

$= 72\{-\sin^2 6t + 1 - \sin^2 6t\}$

$= 72(1 - 2\sin^2 6t)$

$= 72(1-2x)$

$= -144(x - \tfrac{1}{2})$

QUESTION 3.

(a)(i) $\dot{y} = -10t + 40$

$\dot{y} = 0 \Rightarrow 10t = 40$

$t = 4$ s.

$\therefore y = -5(4)^2 + 40(4) + 100$

$= 180$ m

(ii) $y = 0 \Rightarrow -5t^2 + 40t + 100 = 0$

$t^2 - 8t - 20 = 0$

$(t-10)(t+2) = 0$

$t = 10 \quad (t \geq 0)$

$\therefore x = 30(10)$

$= 300$ m.

(b) $\{5, 5, 5, 10, 10, 10, 10, 10, 20, 20\}$

(i) All diff $= {}^3C_1 \cdot {}^5C_1 \cdot {}^2C_1$

$= 30$

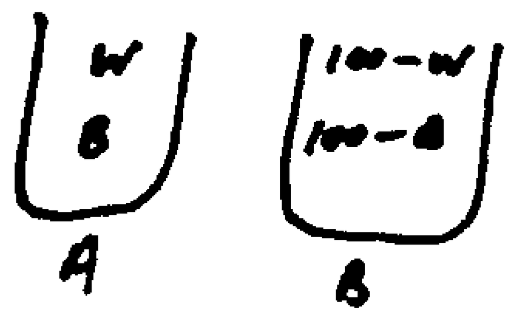
Q3 (b)(ii) ($10, $10, $10) or ($5, $5, $20)

$N^o = {}^4C_3 + {}^4C_2 \cdot {}^2C_1$

$= 16$

QUESTION 4.

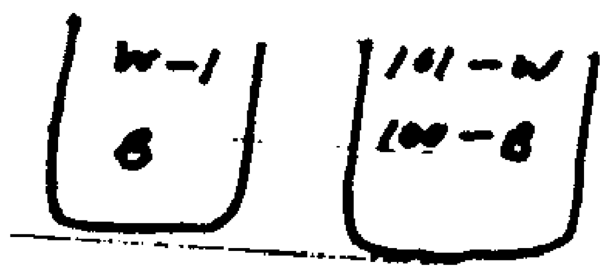
(a) In urn A let W = number of white + B = number of black



$\frac{W}{W+B} = \frac{2}{3}$

$3W = 2W + 2B$

$W = 2B$ ——— ①



$\frac{100-B}{201-W-B} = \frac{2}{3}$

$300 - 3B = 402 - 2W - 2B$

$2W - B = 102$ ——— ②

sub. ① into ②

$4B - B = 102$

$3B = 102$

$B = 34$

$+ \; W = 68.$

$\therefore$ 68 white + 34 black.

(b) (i) period $= 2\pi/n$

$\therefore \frac{2\pi}{n} = 8$

$n = \pi/4$

max speed $= na$

$3\pi = \frac{\pi}{4}a$

$a = 12$

$t = 0, \; x = 6$

$6 = 12\cos\alpha$

$\cos\alpha = \frac{1}{2}$

$\alpha = \pi/3$

or

$x = a\cos\left(\frac{\pi}{4}t + \alpha\right)$

$\dot{x} = -\frac{\pi a}{4}\sin\left(\frac{\pi}{4}t + \alpha\right)$

$\therefore \; 3\pi = -\frac{\pi}{4}a$

$a = -12$

$t = 0$

$+6 = -12\cos\alpha$

$\cos\alpha = -\frac{1}{2}$

$\alpha = \frac{2\pi}{3}$

Q4(b)(ii) $x = 12\cos\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$

$0 = 12\cos\left(\frac{\pi}{4}t + \frac{\pi}{3}\right)$

$\frac{\pi}{4}t + \frac{\pi}{3} = \frac{\pi}{2}$

$\frac{\pi}{4}t = \frac{\pi}{6}$

$t = \frac{2}{3}$ min

or $x = -12\cos\left(\frac{\pi}{4}t + \frac{2\pi}{3}\right)$

$0 = -12\cos\left(\frac{\pi}{4}t + \frac{2\pi}{3}\right)$

$\frac{\pi}{4}t + \frac{2\pi}{3} = \frac{3\pi}{2}$

$\frac{\pi}{4}t = \frac{3\pi}{2} - \frac{2\pi}{3}$

$= \frac{5\pi}{6}$

$t = \frac{5\pi}{6} \cdot \frac{4}{\pi}$

$= 3\frac{1}{3}$ min

QUESTION 5

(a) (i) N^o arrangements $= 7!$

(ii) arrange girls $= 4!$

arrange girl group + 4 boys $= 4!$

Total $= 4!4!$

$\therefore$ Prob $= \frac{4!4!}{7!}$

(iii) place girls $= 3!$

place boys $= 4!$

Total $= 3!4!$

Prob $= \frac{3!4!}{7!}$

(b) (i) $t = 0, \quad N = \frac{10^6}{100 + 1900}$

$= 500$

(ii) $t \to \infty, \quad e^{-0.05t} \to 0$

$\therefore N \to \frac{10^6}{100}$

$\therefore N = 10\,000$

Q5 (iii) $N = 10^6(100 + 1900e^{-0.05t})^{-1}$

$\frac{dN}{dt} = 10^6 \cdot -1(100 + 1900e^{-0.05t})^{-2} \cdot -0.05 \times 1900e^{-0.05t}$

$= \frac{10^6 \times 0.05 \times 1900 \; e^{-0.05t}}{(100 + 1900e^{-0.05t})^2}$

$= \frac{95 \times 10^6 e^{-0.05t}}{(100 + 1900e^{-0.05t})^2}$

Now

$\frac{N}{10^6} = \frac{1}{100 + 1900e^{-0.05t}}$ —— ①

& $100 + 1900e^{-0.05t} = \frac{10^6}{N}$

$1900e^{-0.05t} = \frac{10^6}{N} - 100$

$e^{-0.05t} = \frac{10^6 - 10^2 N}{1900N}$ —— ② $= \frac{10^4 - N}{19N}$

$\therefore \frac{dN}{dt} = 95 \times 10^6 \left(\frac{10^4 - N}{19N}\right) \cdot \left(\frac{N}{10^6}\right)^2$

$= 95 \times 10^6 \frac{(10000 - N)}{19N} \cdot \frac{N^2}{10^{12}}$

$= \frac{95(10000 - N)N}{19 \times 10^6}$

$= \frac{N(10000 - N)}{200000}$

(iv) rate greatest when $N = 5000$

QUESTION 6

(a) let $u = x^2/16$

$\therefore \frac{d}{dx}\left(\cos^{-1}\frac{x^2}{16}\right) = \frac{d}{du}(\cos^{-1}u) \cdot \frac{du}{dx}$

$= \frac{-1}{\sqrt{1-u^2}} \cdot \frac{2x}{16}$

$= \frac{-x}{8\sqrt{1 - \frac{x^4}{256}}}$

Q6(b)(iii) (cont)

$\frac{dx}{dt} = \frac{-2\sqrt{256 - x^4}}{x}$

$\frac{dt}{dx} = \frac{-x}{2\sqrt{256 - x^4}}$

$t = \int \frac{-x}{2\sqrt{256 - x^4}} \, dx$

$= \frac{1}{4} \int \frac{-2x}{\sqrt{256 - x^4}} \, dx$

$t = \frac{1}{4}\cos^{-1}\left(\frac{x^2}{16}\right) + c$

$t = 0, x = 4 \Rightarrow 0 = \frac{1}{4}\cos^{-1}(1) + c$

$\therefore c = 0$

$\therefore t = \frac{1}{4}\cos^{-1}\left(\frac{x^2}{16}\right)$.

(iv) $x = 2\sqrt{2}$, $t = \frac{1}{4}\cos^{-1}\left(8/16\right)$

$= \frac{1}{4}\cos^{-1}(1/2)$

$= 1/4 \cdot \pi/3$

$t = \pi/12$

≠